Today

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Finish Euclid.

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Bijection/CRT/Isomorphism.

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Fermat's Little Theorem.

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Review for Midterm.

Finding an inverse?

We showed how to efficiently tell if there is an inverse.

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Extend euclid to find inverse.

Euclid's GCD algorithm.

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(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
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Computes the gcd(x, y) in $O(n)$ divisions. (Remember $n = \log_2 x$.)

For x and m, if $\gcd(x, m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

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How do we **find** a multiplicative inverse?

Extended GCD

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$$ax + by$$

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"Make d out of sum of multiples of x and y."

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What is multiplicative inverse of x modulo m?

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$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$(3)12 + (-1)35 = 1$.

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.

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So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.
The multiplicative inverse of $12 \pmod{35}$ is 3.

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

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$a = 3$ and $b = -1$.

The multiplicative inverse of $12 \pmod{35}$ is 3.

Check: 3(12)

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So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$(3)12 + (-1)35 = 1$.

$a = 3$ and $b = -1$.
The multiplicative inverse of $12 \pmod{35}$ is 3.

Check: $3(12) = 36$

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Euclid's Extended GCD Theorem: For any x, y there are integers a, b such that
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"Make d out of sum of multiples of x and y."

What is multiplicative inverse of x modulo m?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$(3)12 + (-1)35 = 1.$

$a = 3$ and $b = -1$.
The multiplicative inverse of $12 \pmod{35}$ is 3.

Check: $3(12) = 36 = 1 \pmod{35}$.

Make d out of multiples of x and y..?

```
gcd(35,12)
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
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        1
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How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11$

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12)$
Get 11 from 35 and 12 and plugin....

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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        1
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How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify.

Make *d* out of multiples of *x* and *y*..?

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  gcd(12, 11)  ;;  gcd(12, 35%12)
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$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify.

Make d out of multiples of x and y..?

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  gcd(12, 11)  ;;  gcd(12, 35%12)
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How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$.

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
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Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax+by$.

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Example:

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ext-gcd(35,12)
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Example:

```
ext-gcd(35,12)
  ext-gcd(12, 11)
```

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ext-gcd(x,y)
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Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b =$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
```

Extended GCD Algorithm.

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ext-gcd(x,y)
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         (d, a, b) := ext-gcd(y, mod(x,y))
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```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 11/1 \rfloor \cdot 0 = 1$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
```

Extended GCD Algorithm.

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ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = \quad 0 - \lfloor 12/11 \rfloor \cdot 1 = -1$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
  return (1,1,-1)    ;; 1 = (1)12 + (-1)11
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b =$ $1 - \lfloor 35/12 \rfloor \cdot (-1) = 3$

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

Extended GCD Algorithm.

```
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  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
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Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example:

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
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     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Theorem: Returns (d,a,b), where $d = gcd(a,b)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.[1]

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x,0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

[1] Assume d is $gcd(x,y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x,0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$

[1] Assume d is $gcd(x,y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x,0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**$(y, \mod(x,y))$ returns (d, a, b) with
$d = ay + b(\mod(x,y))$

[1] Assume d is $gcd(x,y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd($x,0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**($y, \mod(x,y)$) returns (d, a, b) with

$$d = ay + b(\mod(x,y))$$

ext-gcd(x, y) calls **ext-gcd**($y, \mod(x,y)$) so

[1] Assume d is $gcd(x,y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x, 0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
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Confirm: $-119 + 120 = 1$

Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

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Internet Security: Next Week!

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A bit slow for large values.

Simple Chinese Remainder Theorem.

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Consider $u = n(n^{-1} \pmod{m})$.
$u = 0 \pmod{n}$ $\qquad$ $u = 1 \pmod{m}$

Consider $v = m(m^{-1} \pmod{n})$.

Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m,n)=1.

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Let $x = au + bv$.

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Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$

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Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

Only solution?

Simple Chinese Remainder Theorem.

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Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

Only solution? If not, two solutions, x and y.

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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

Only solution? If not, two solutions, x and y.
$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.

Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

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Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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Only solution? If not, two solutions, x and y.
$(x-y) \equiv 0 \pmod{m}$ and $(x-y) \equiv 0 \pmod{n}$.
$\Longrightarrow$ $(x-y)$ is multiple of m and n since gcd(m,n)=1.

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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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Only solution? If not, two solutions, x and y.
$(x-y) \equiv 0 \pmod{m}$ and $(x-y) \equiv 0 \pmod{n}$.
$\implies (x-y)$ is multiple of m and n since gcd(m,n)=1.
$\implies x - y \geq mn$

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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.
$\implies (x - y)$ is multiple of m and n since gcd(m,n)=1.
$\implies x - y \geq mn \implies x, y \notin \{0, \ldots, mn - 1\}$.
Thus, only one solution modulo mn.

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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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Fermat's Theorem: Reducing Exponents.

Fermat's Little Theorem: For prime p, and $a \not\equiv 0 \pmod{p}$,

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All different modulo p since a has an inverse modulo p.

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S contains representative of $\{1, \ldots, p-1\}$ modulo p.

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□

Fermat and Exponent reducing.

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What is $2^{101} \pmod{7}$?

Fermat and Exponent reducing.

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What is $2^{101} \pmod{7}$?

Wrong: $2^{101} = 2^{7*14+3} = 2^3 \pmod{7}$

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Correct: $2^{101} = 2^{6*16+5} = 2^5 = 32 = 4 \pmod{7}$.

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Correct: $2^{101} = 2^{6*16+5} = 2^5 = 32 = 4 \pmod{7}$.

For a prime modulus, we can reduce exponents modulo $p-1$!

Midterm Review

Now...

First there was logic...

A statement is true or false.

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Statements?

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A statement is true or false.

Statements?

$3 = 4 - 1$?

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$?

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

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First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

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3 ?

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

$n = 3$?

First there was logic...

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Statements?

$3 = 4 - 1$? Statement!

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3 ? Not a statement!

$n = 3$? Not a statement...

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

$n = 3$? Not a statement...but a predicate.

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

$n = 3$? Not a statement...but a predicate.

Predicate: Statement with free variable(s).

First there was logic...

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Statements?

$3 = 4 - 1$? Statement!

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Predicate: Statement with free variable(s).

Example: $x = 3$

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Statements?

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Predicate: Statement with free variable(s).

Example: $x = 3$

Given a value for x, becomes a statement.

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A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

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Predicate: Statement with free variable(s).

Example: $x = 3$

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Predicate?

$n > 3$?

First there was logic...

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Statements?

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Predicate: Statement with free variable(s).

Example: $x = 3$

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Predicate?

$n > 3$? Predicate: $P(n)$!

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A statement is true or false.

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Example: $x = 3$

Given a value for x, becomes a statement.

Predicate?

$n > 3$? Predicate: $P(n)$!

$x = y$?

First there was logic...

A statement is true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

$n = 3$? Not a statement...but a predicate.

Predicate: Statement with free variable(s).

Example: $x = 3$

Given a value for x, becomes a statement.

Predicate?

$n > 3$? Predicate: $P(n)$!

$x = y$? Predicate: $P(x, y)$!

First there was logic...

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Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

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You got this!

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Propositional Expressions and Logical Equivalence

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$$(A \implies B) \equiv (\neg A \vee B)$$
$$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$$

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

Connecting Statements

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You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
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Proofs: truth table or manipulation of known formulas.

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

Proofs: truth table or manipulation of known formulas.

$(\forall x)(P(x) \wedge Q(x)) \equiv (\forall x)P(x) \wedge (\forall x)Q(x)$

..and then proofs...

Direct: $P \implies Q$

..and then proofs...

Direct: $P \implies Q$

Example: a is even $\implies a^2$ is even.

..and then proofs...

Direct: $P \implies Q$

Example: a is even $\implies a^2$ is even.

Approach: What is even?

..and then proofs...

Direct: $P \implies Q$

Example: a is even $\implies a^2$ is even.

Approach: What is even? $a = 2k$

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$a^2 = 2(2k^2)$

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Integers closed under multiplication!

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Contrapositive: $P \implies Q$

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Contrapositive: $P \implies Q$ or $\neg Q \implies \neg P$.

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Contrapositive: a is even $\implies a^2$ is even.

Contradiction: P

$\neg P \implies$ **false**

..and then proofs...

Direct: $P \implies Q$

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Example: a^2 is odd $\implies a$ is odd.

Contrapositive: a is even $\implies a^2$ is even.

Contradiction: P

$\neg P \implies$ **false**

$\neg P \implies R \wedge \neg R$

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Useful for prove something does not exist:

..and then proofs...

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$

..and then proofs...

Direct: $P \implies Q$

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$a^2 = 4k^2$.

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Contrapositive: a is even $\implies a^2$ is even.

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$\neg P \implies R \wedge \neg R$

Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

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Example: a is even $\implies$ a^2 is even.

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$a^2 = 4k^2$.

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

Example: finite set of primes

..and then proofs...

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

Example: finite set of primes does not exist.

Example: rogue couple does not exist.

...jumping forward..

Contradiction in induction:

...jumping forward..

Contradiction in induction:
contradict place where induction step doesn't hold.

...jumping forward..

Contradiction in induction:
contradict place where induction step doesn't hold.

Well Ordering Principle.

...jumping forward..

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Well Ordering Principle.
Stable Marriage:

...jumping forward..

Contradiction in induction:
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Stable Marriage:
first day where women does not improve.

...jumping forward..

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first day where women does not improve.
first day where any man rejected by optimal women.

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...and then induction...

$$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n)$.

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

...and then induction...

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Induction on n.

...and then induction...

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Induction on n.

Base: $8|3^2 - 1$.

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Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.

...and then induction...

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Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.

Induction Step: Prove $P(n+1)$

$3^{2n+2} - 1 =$

...and then induction...

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Thm: For all $n \geq 1$, $8|3^{2n}-1$.

Induction on n.

Base: $8|3^2-1$.

Induction Hypothesis: Assume $P(n)$: True for some n.

Induction Step: Prove $P(n+1)$

$3^{2n+2}-1 = 9(3^{2n})-1$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n)$.

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$3^{2n+2} - 1 = 9(3^{2n}) - 1$ (by induction hypothesis)

...and then induction...

$P(0) \land ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

Thm: For all $n \geq 1$, $8|3^{2n}-1$.

Induction on n.

Base: $8|3^2-1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n}-1 = 8d)$

Induction Step: Prove $P(n+1)$

$$\begin{aligned} 3^{2n+2}-1 &= 9(3^{2n})-1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1)-1 \end{aligned}$$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$$\begin{aligned} 3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1) - 1 \\ &= 72d + 8 \end{aligned}$$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

Thm: For all $n \geq 1$, $8|3^{2n} - 1$.

Induction on n.

Base: $8|3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$$\begin{aligned} 3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1) - 1 \\ &= 72d + 8 \\ &= 8(9d+1) \end{aligned}$$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n)$.

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
($3^{2n} - 1 = 8d$)

Induction Step: Prove $P(n+1)$

$$\begin{aligned} 3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1) - 1 \\ &= 72d + 8 \\ &= 8(9d+1) \end{aligned}$$

Divisible by 8.

...and then induction...

$$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$$

Thm: For all $n \geq 1$, $8|3^{2n} - 1$.

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□

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n-men, n-women.

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No, for roommates problem.

TMA.

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Not rogue couple!

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Optimal partner if best partner in any stable pairing.

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Man optimal $\Longrightarrow$ Woman pessimal.

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 Woman optimal $\implies$ Man pessimal.

...Graphs...

$$G = (V, E)$$

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Connected Graph: one connected component.

Graph Algorithm: Eulerian Tour

Thm: Every connected graph where every vertex has even degree has an Eulerian Tour; a tour which visits every edge exactly once.

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Recurse on connected components.

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Put together.

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Property: walk visits every component.

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Property: walk visits every component.

Proof Idea: Original graph connected.

Graph Algorithm: Eulerian Tour

Thm: Every connected graph where every vertex has even degree has an Eulerian Tour; a tour which visits every edge exactly once.

Algorithm:

Take a walk using each edge at most once.

Property: return to starting point.

Proof Idea: Even degree.

Recurse on connected components.

Put together.

Property: walk visits every component.

Proof Idea: Original graph connected.

Graph Coloring.

Given $G = (V, E)$, a coloring of a G assigns colors to vertices V where for each edge the endpoints have different colors.

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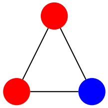

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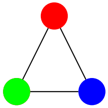

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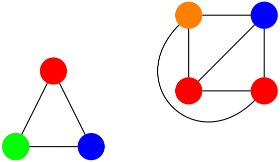

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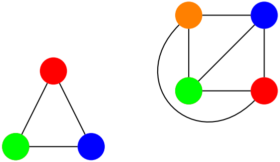

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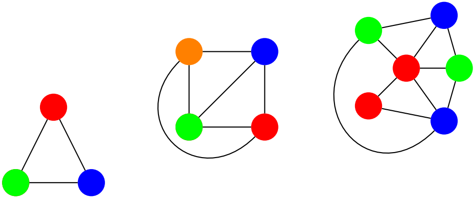

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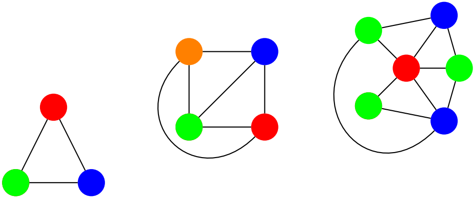

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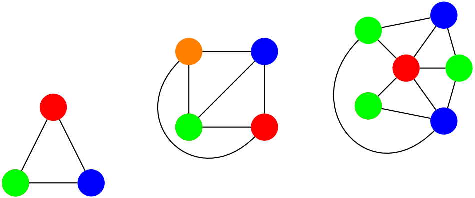

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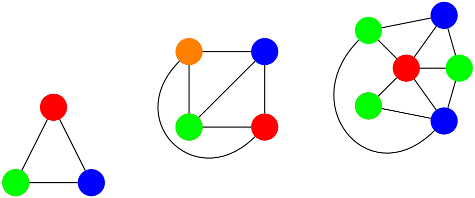

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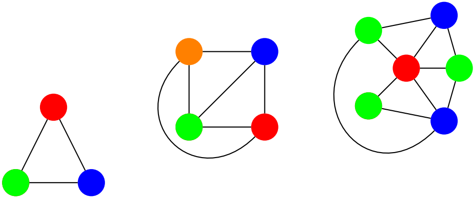

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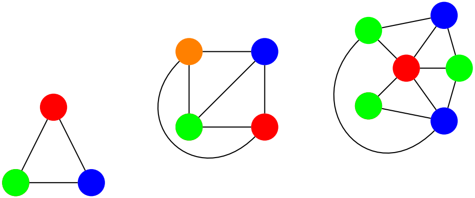

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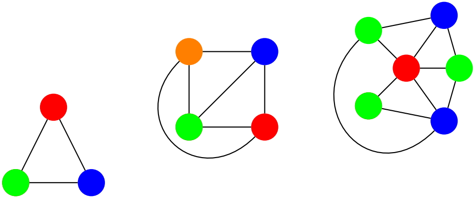

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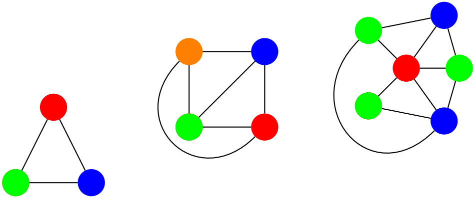

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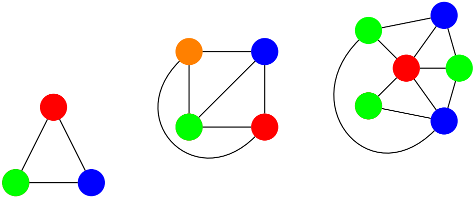

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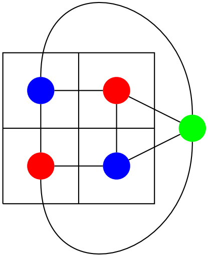
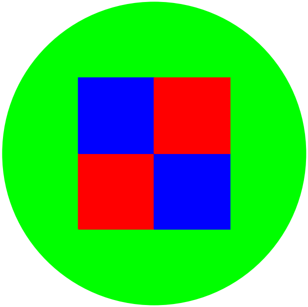
Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





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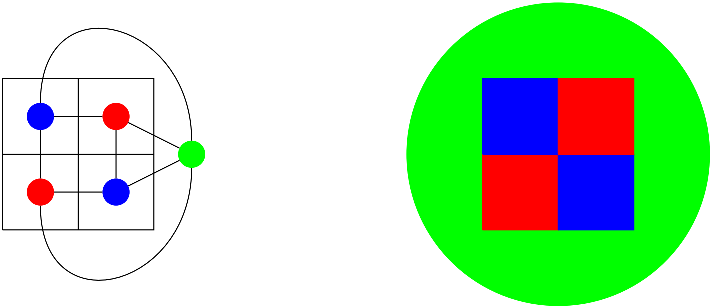

Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

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Five color theorem: summary.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

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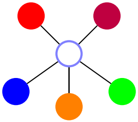

Either switch green.

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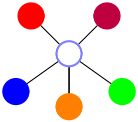

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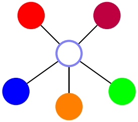

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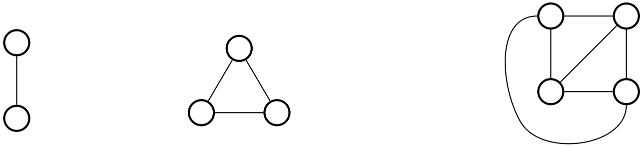
One will work.

Graph Types: Complete Graph.



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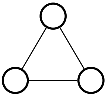

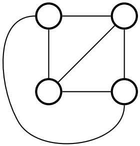

K_n, $|V| = n$

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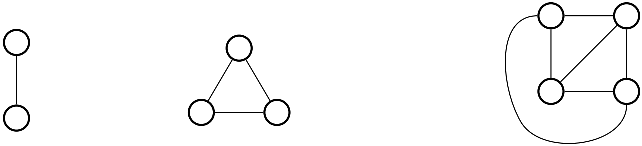

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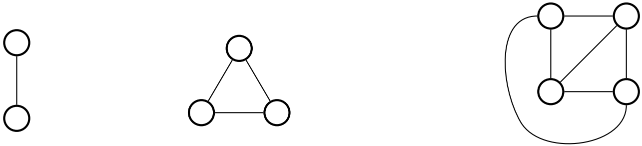

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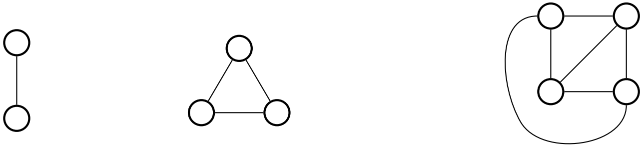

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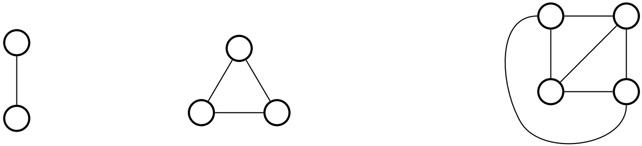

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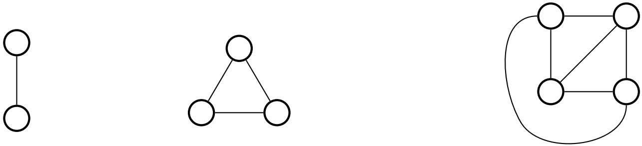

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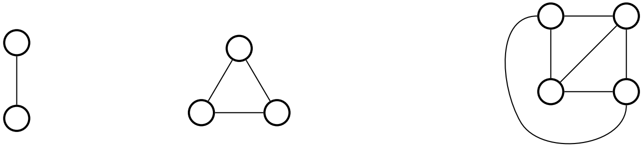

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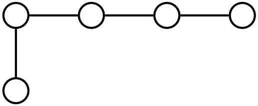
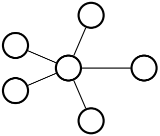
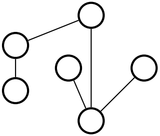
Trees.







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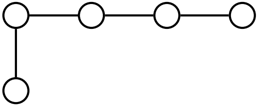

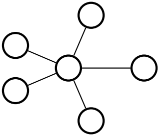

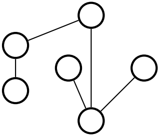

Definitions:

A connected graph without a cycle.

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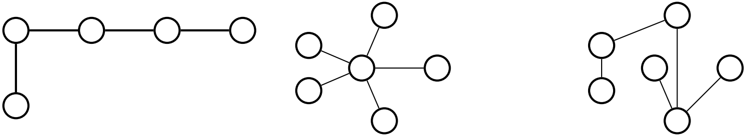

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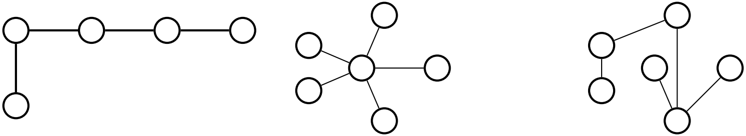

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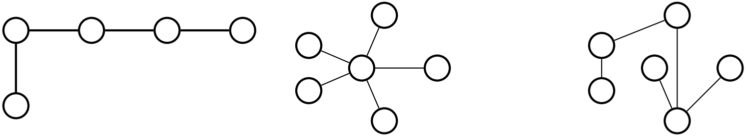

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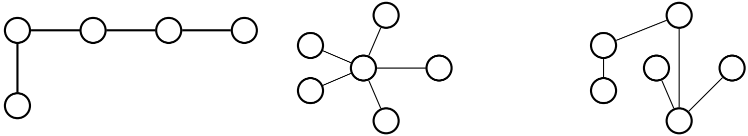

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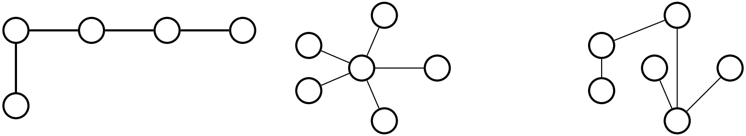

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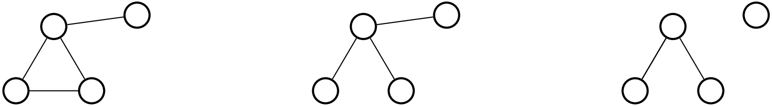

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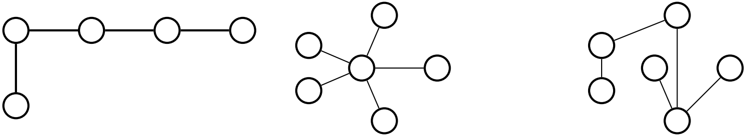

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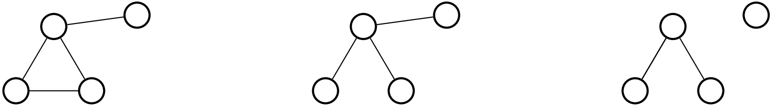

Minimally connected, minimum number of edges to connect.

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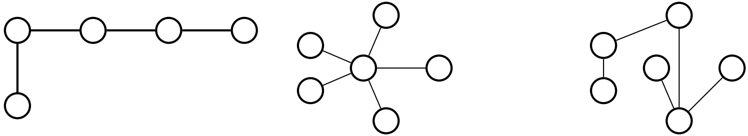

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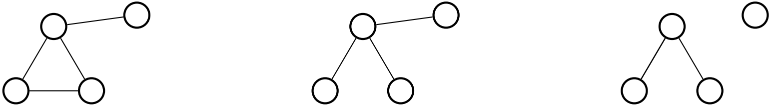
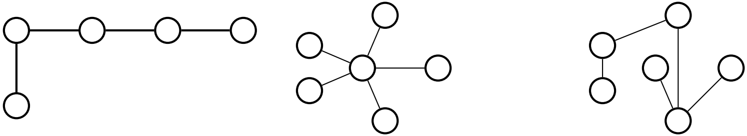


Minimally connected, minimum number of edges to connect.

Property:

Trees.



Definitions:

A connected graph without a cycle.
A connected graph with $|V|-1$ edges.
A connected graph where any edge removal disconnects it.
An acyclic graph where any edge addition creates a cycle.

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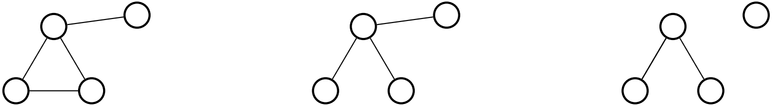

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Property:
Can remove a single node and break into components of size at most $|V|/2$.

Hypercube

Hypercubes.

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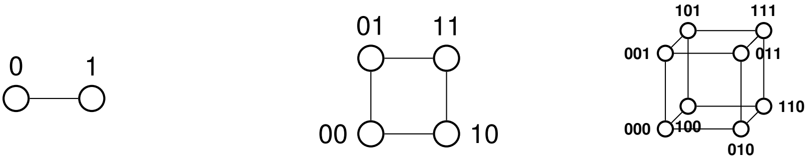

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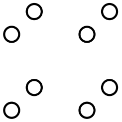

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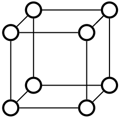

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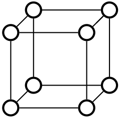

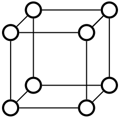

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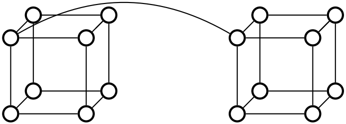

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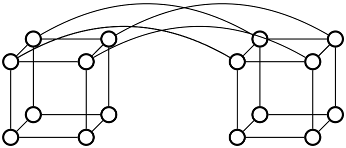

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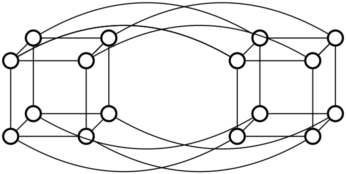

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Extended-gcd(x,y)

Modular Arithmetic Inverses and GCD

x has inverse modulo m if and only if $gcd(x,m) = 1$.

Group structures more generally.

Proof Idea:
$\{0x, \ldots, (m-1)x\}$ are distinct modulo m if and only if $gcd(x,m) = 1$.

Finding gcd.
$gcd(x,y) = gcd(y, x-y) = gcd(y, x \pmod y)$.

Give recursive Algorithm! Base Case? $gcd(x,0) = x$.

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by adding and subtracting multiples of x and y

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Hand calculation: egcd.

Extended GCD: egcd$(7,60) = 1$.

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$$7(0) + 60(1) \quad = \quad 60$$

Hand calculation: egcd.

Extended GCD: $\text{egcd}(7,60) = 1$.
egcd(7,60).

$$\begin{aligned} 7(0)+60(1) &= 60 \\ 7(1)+60(0) &= 7 \end{aligned}$$

Hand calculation: egcd.

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egcd(7,60).

$$\begin{aligned} 7(0)+60(1) &= 60 \\ 7(1)+60(0) &= 7 \\ 7(-8)+60(1) &= 4 \end{aligned}$$

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7(0)+60(1) &= 60 \\
7(1)+60(0) &= 7 \\
7(-8)+60(1) &= 4 \\
7(9)+60(-1) &= 3
\end{aligned}
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$$\begin{aligned} 7(0)+60(1) &= 60 \\ 7(1)+60(0) &= 7 \\ 7(-8)+60(1) &= 4 \\ 7(9)+60(-1) &= 3 \\ 7(-17)+60(2) &= 1 \end{aligned}$$

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Confirm:

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Confirm: $-119+120=1$

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Confirm: $-119 + 120 = 1$

$d = e^{-1} = -17 = 43 = \pmod{60}$

Midterm format

Time: 120 minutes.

Midterm format

Time: 120 minutes.

Some short answers.

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.

Midterm format

Time: 120 minutes.

Some short answers.

Get at ideas that you learned.

Know material well:

Midterm format

Time: 120 minutes.

Some short answers.

Get at ideas that you learned.

Know material well: fast,

Midterm format

Time: 120 minutes.

Some short answers.
 Get at ideas that you learned.
 Know material well: fast, correct.

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.
Know material well: fast, correct.
Know material medium:

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.
Know material well: fast, correct.
Know material medium: slower,

Midterm format

Time: 120 minutes.

Some short answers.

Get at ideas that you learned.

Know material well: fast, correct.

Know material medium: slower, less correct.

Midterm format

Time: 120 minutes.

Some short answers.
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Know material not so well:	

Midterm format

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Proofs, algorithms,

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  Proofs, algorithms, properties.
  Not so much calculation.

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See piazza for more resources.

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See piazza for more resources.
E.g., TA videos for past exams.

Wrapup.

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