Today

Finish Euclid.
Bijection/CRT/Isomorphism.
Fermat’s Little Theorem.
Review for Midterm.

Finding an inverse?

We showed how to efficiently tell if there is an inverse.
Extend euclid to find inverse.

Extended GCD

Euclid’s Extended GCD Theorem: For any \( x, y \) there are integers \( a, b \) such that
\[
ax + by = d
\]
where \( d = \gcd(x, y) \).

“Make \( d \) out of sum of multiples of \( x \) and \( y \).”

What is multiplicative inverse of \( x \) modulo \( m \)?

By extended GCD theorem, when \( \gcd(x, m) = 1 \).
\[
ax = 1 \mod m
\]

So \( a \) multiplicative inverse of \( x \) (mod \( m \))!!

Example. For \( x = 12 \) and \( y = 35 \), \( \gcd(12, 35) = 1 \).
\[
(3)12 + (-1)35 = 1
\]

The multiplicative inverse of 12 (mod 35) is 3.
Check: \( 3(12) = 36 \equiv 1 \pmod{35} \).

Make \( d \) out of multiples of \( x \) and \( y \).?

\[
\gcd(35, 12) \quad \gcd(12, 11) \quad \gcd(11, 1) \quad \gcd(1, 0)
\]

How did gcd get 11 from 35 and 12?

\[
35 - (\frac{35}{12})12 = 35 - (2)12 = 11
\]

How does gcd get 1 from 12 and 11?

\[
12 - (\frac{12}{11})11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?
Get 1 from 12 and 11.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35
\]

Get 11 from 35 and 12 and plug in... Simplify. \( a = 3 \) and \( b = -1 \).
**Extended GCD Algorithm.**

```plaintext
ext-gcd(x, y)
  if y = 0 then return(x, 1, 0)
  else
    (d, a, b) := ext-gcd(y, mod(x, y))
    return (d, b, a − floor(x/y) * b)
```

Claim: Returns $(d, a, b) = gcd(a, b)$ and $d = ax + by$.

**Example:** $x = 1234567890987654321$, $y = 9876543210987654321$,

```
ext-gcd(1234567890987654321, 9876543210987654321)
```

**Theorem:** Returns $(d, a, b)$, where $d = gcd(a, b)$ and $d = ax + by$.

```plaintext
ext-gcd(x, y)
  if y = 0 then return(x, 1, 0)
  else
    (d, a, b) := ext-gcd(y, mod(x, y))
    return (d, b, a − floor(x/y) * b)
```

**Correctness.**

**Proof:** Strong Induction.1

**Base:** $ext-gcd(x, 0) = (d = x, 1, 0)$ with $x = (1)x + (0)y$.

**Induction Step:** Returns $(d, A, B)$ with $d = Ax + By$

1 Assume $d = gcd(x, y)$ by previous proof.

Ext-GCD($x, y$) calls Ext-GCD($y, mod(x, y)$) so

$$d = ay + b (\text{mod} (x, y))$$

$$= ay + b (x - x/y)$$

$$= bx + (a - x/y) b y$$

And ext-gcd returns $(d, a, b(x - [x/y] b))$ so theorem holds!

**Hand Calculation Method for Inverses.**

Example: $gcd(7, 60) = 1$.

```
egcd(7, 60):
```

```
7(0) + 60(1)  =  60
7(1) + 60(0)  =  7
7(−8) + 60(1) =  4
7(9) + 60(−1) =  3
7(−17) + 60(2) =  1
```

Confirm: $−119 + 120 = 1$

**Conclusion:** Can find multiplicative inverses in $O(n)$ time!

Very different from elementary school: try 1, try 2, try 3...

$2^n$ divisions.

Inverse of 500, 000, 357 modulo 1,000, 000, 000, 000?

≤ 80 divisions.

versus 1,000,000

Internet Security.
Public Key Cryptography: 512 digits.

512 divisions vs.

$\underbrace{10000000000000000000000000000000000000000000}^d$ divisions.

Internet Security: Next Week!
Bijections

Bijection is one to one and onto.

Bijection:
\[ f : A \rightarrow B. \]
Domain: A, Co-Domain: B.

Versus Range.
E.g. \( \sin(x) \).

Range is \([-1, 1]\). Onto: \([-1, 1]\).

Not one-to-one. \( \sin(\pi) = \sin(0) = 0 \).

Range Definition is always onto.

Find \( f(x) = ax \mod m. \)
\( f : \{0, \ldots, m - 1\} \rightarrow \{0, \ldots, m - 1\}. \)

Domain/Co-Domain: \([0, m - 1]\).

When is it a bijection?
When \( \gcd(a,m) \) is ...? ... 1.

Not Example: \( a = 2, m = 4, f(0) = f(2) = 0 \mod 4 \).

Lots of Mods

\( x = 5 \mod 7 \) and \( x = 3 \mod 5 \).

What is \( x \mod 35 \)?

Let’s try 5. Not 3 \( \mod 5 \! \). Let’s try 3. Not 5 \( \mod 7 \! \).

If \( x = 5 \mod 7 \) then \( x \) is in \( \{5, 12, 19, 26, 33\} \).

Oh, only 33 is \( 3 \mod 5 \).

Hmm... only one solution.
A bit slow for large values.

Fermat's Theorem: Reducing Exponents.

Fermat’s Little Theorem: For prime \( p \), and \( a \neq 0 \mod (p) \),
\[ a^{p-1} \equiv 1 \mod (p) . \]

Proof: Consider \( S = \{a, 2, \ldots, a - (p-1)\} \).
All different modulo \( p \) since \( a \) has an inverse modulo \( p \).
\( S \) contains representative of \( \{1, \ldots, p-1\} \) modulo \( p \).

\[ (a - 1) \cdot (a - 2) \cdot \cdots \cdot (a - (p-1)) \equiv 1 \cdot 2 \cdot \cdots \cdot (p-1) \mod p. \]

Since multiplication is commutative.

\[ a^{p-1} \equiv 1 \cdot (p-1) \mod p. \]

Each of \( 2 \cdot \cdots \cdot (p-1) \) has an inverse modulo \( p \), solve to get...
\[ a^{p-1} \equiv 1 \mod p. \]

Fermat and Exponent reducing.

Fermat’s Little Theorem: For prime \( p \), and \( a \neq 0 \mod (p) \),
\[ a^{p-1} \equiv 1 \mod (p) . \]

What is \( 2^{101} \mod 7 \)?

Wrong: \( 2^{101} = 2^{7 \cdot 14 + 3} = 2^3 \mod 7 \)

Fermat: 2 is relatively prime to 7. \( \rightarrow 2^6 = 1 \mod 7 \).

Correct: \( 2^{101} = 2^{6 \cdot 16 + 5} = 2^5 = 32 \equiv 4 \mod 7 \).

For a prime modulus, we can reduce exponents modulo \( p - 1 \! \).
First there was logic...

A statement is true or false.

Statements?

3 \neq 4 \rightarrow 1 \text{ ? Statement!}

3 = 5 \text{ ? Statement!}

3 ? Not a statement!

n = 3 \text{ ? Not a statement...but a predicate.}

Predicate: Statement with free variable(s).

Example: \(x = 3\)

Given a value for \(x\), becomes a statement.

Predicate?

\(n = 3\) ? Predicate: \(P(n)\)

\(x = y\) ? Predicate: \(P(x, y)\)!

\(x = y\)? No. An expression, not a statement.

Quantifiers:

\(\forall x \ P(x)\): For every \(x\), \(P(x)\) is true.

\(\exists x \ P(x)\): There exists an \(x\), where \(P(x)\) is true.

\(\forall n \in N, n^2 \geq n\).

\(\forall x \in R, (y \in R) y > x\).

...jumping forward..

Contradiction in induction:

contradict place where induction step doesn’t hold.

Well Ordering Principle.

Stable Marriage:

first day where women does not improve.

does not exist.

Connecting Statements

\(A \land B, A \lor B, \neg A\).

You got this!

Propositional Expressions and Logical Equivalence

\((A \rightarrow B) \equiv (\neg A \lor B)\)

\((A \lor B) \equiv (\neg A \land \neg B)\)

Proofs: truth table or manipulation of known formulas.

\((\forall x)(P(x) \land Q(x)) \equiv (\forall x)P(x) \land (\forall x)Q(x)\)

...and then induction...

\(P(0) \land ((\forall n)(P(n) \rightarrow P(n + 1)) \equiv (\forall n \in N) P(n)\).

Thm: For all \(n \geq 1\), \(8|3^{2n} − 1\).

Induction on \(n\).

Base: \(8|3^2 − 1\).

Induction Hypothesis: Assume \(P(n)\): True for some \(n\).

\((3^{2n} − 1 = 8d)\)

Induction Step: Prove \(P(n + 1)\)

\(3^{2n+2} − 1 = 9(3^{2n}) − 1 \text{ (by induction hypothesis)}\)

\(= 9(8d + 1) − 1\)

\(= 72d + 8\)

\(= 8(9d + 1)\)

Divisible by 8.

Stable Marriage: a study in definitions and WOP.

n-men, n-women.

Each person has completely ordered preference list contains every person of opposite gender.

Pairing.

Set of pairs \((m, w)\) containing all people exactly once.

How many pairs? \(n\).

People in pair are partners in pairing.

Rogue Couple in a pairing.

A man and woman, who like each other more than their partners

Stable Pairing.

Pairing with no rogue couples.

Does stable pairing exist?

No, for roommates problem.

..and then proofs...

Direct: \(P \rightarrow Q\)

Example: a is even \(\rightarrow a^2 \text{ is even.}\)

Approach: What is even? \(a = 2k\)

\(a^2 = 4k^2\).

What is even?

\(a^2 = 2(2k^2)\)

Integers closed under multiplication!

\(a^2 \text{ is even.}\)

Contrapositive: \(P \rightarrow Q \text{ or } \neg Q \rightarrow \neg P\).

Example: \(a^2 \text{ is odd} \rightarrow a \text{ is odd.}\)

Contrapositive: \(a \text{ is even} \rightarrow a^2 \text{ is even.}\)

Contradiction: \(P \rightarrow Q \text{ or } \neg Q \rightarrow \neg P\)

Useful for prove something does not exist:

Example: rational representation of \(\sqrt{2}\) does not exist.

Example: finite set of primes does not exist.

Example: rogue couple does not exist.
TMA. Traditional Marriage Algorithm:

Each Day:
- All men propose to favorite non-rejecting woman.
- Every woman rejects all but best men who propose.

Useful Algorithmic Definitions:
- Man crosses off woman who rejected him.
- Woman's current proposer is "on string."

"Propose and Reject." : Either men propose or women. But not both. Traditional propose and reject where men propose.

Key Property: Improvement Lemma:
- Every day, if man on string for woman, ⇒ any future man on string is better.

Stability:
- No rogue couple.

Thm: TMA produces male optimal pairing, S.
- First man M to lose optimal partner.
- Better partner W for M.
- Different stable pairing T.
- There is M' who bumps M in TMA.
- W prefers M'.
- M' and W is rogue couple in T.

Thm: woman pessimal.
- Man optimal ⇒ Woman pessimal.
- Woman optimal ⇒ Man pessimal.

Optimality/Pessimal

Optimal partner if best partner in any stable pairing.
- Not necessarily first in list.
- Possibly no stable pairing with that partner.
- Man-optimal pairing is pairing where every man gets optimal partner.

Thm:
- G = (V,E)
- V - set of vertices.
- E ⊆ V × V - set of edges.
- Directed: ordered pair of vertices.
- Adjacent, Incident, Degree.
- In-degree, Out-degree.

Thm: Sum of degrees is 2|E|.
- Edge is incident to 2 vertices.
- Degree of vertices is total incidences.
- Pair of Vertices are Connected:
- If there is a path between them.
- Connected Component: maximal set of connected vertices.
- Connected Graph: one connected component.

Thm: Every connected graph where every vertex has even degree has an Eulerian Tour; a tour which visits every edge exactly once.

Algorithm:
- Take a walk using each edge at most once.
- Property: return to starting point.
- Proof Idea: Even degree.
- Recurse on connected components.
- Put together.
- Property: walk visits every component.
- Proof Idea: Original graph connected.

Graph Coloring.

Given G = (V,E), a coloring of G assigns colors to vertices V where for each edge the endpoints have different colors.

Notice that the last one, has one three colors.
- Fewer colors than number of vertices.
- Fewer colors than max degree node.

Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring = map coloring.

Four color theorem is about planar graphs!
Six color theorem.

Theorem: Every planar graph can be colored with six colors.

Proof:
Recall: \( e \leq 3v - 6 \) for any planar graph where \( v > 2 \).
From Euler's Formula.
Total degree: \( 2e \)
Average degree: \( \leq \frac{2e}{v} \)
There exists a vertex with degree \( < 6 \) or at most 5.
Remove vertex \( v \) of degree at most 5.
Inductively color remaining graph.
Color is available for \( v \) since only five neighbors...
and only five colors are used.

Five color theorem: summary.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Theorem: Every planar graph can be colored with five colors.

Proof: Again with the degree 5 vertex. Again recurse.
Either switch green.
Or try switching orange.
One will work.

Graph Types: Complete Graph.

\( K_n, |V| = n \)
every edge present.
degree of vertex? \( |V| - 1 \).
Very connected.
Lots of edges: \( n(n - 1)/2 \).

Trees.

Definitions:
A connected graph without a cycle.
A connected graph with \( |V| - 1 \) edges.
A connected graph where any edge removal disconnects it.
An acyclic graph where any edge addition creates a cycle.
To tree or not to tree!
Minimally connected, minimum number of edges to connect.
Property:
Can remove a single node and break into components of size at most \( |V|/2 \).

Hypercube

Hypercubes. Really connected. \(|V| \log |V| \) edges!
Also represents bit-strings nicely.
\( G = (V,E) \)
\(|V| = (0,1)^n\).
\(|E| = \{(x,y) | x \text{ and } y \text{ differ in one bit position.}\}\)

Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.
An \( n \)-dimensional hypercube consists of a 0-subcube (1-subcube) which is a \( n-1 \)-dimensional hypercube with nodes labelled 0x (1x) with the additional edges (0x, 1x).
Hypercube: properties

Rudrata Cycle: cycle that visits every node.
Eulerian? If \( n \) is even.
Large Cuts: Cutting off \( k \) nodes needs \( \geq k \) edges.
Best cut? Cut apart subcubes: cuts off \( 2^n \) nodes with \( 2^{n-1} \) edges.
FYI: Also cuts represent boolean functions.

Nice Paths between nodes.
Get from 000100 to 101000.
000100 \( \rightarrow \) 100100 \( \rightarrow \) 101100 \( \rightarrow \) 101000
Correct bits in string, moves along path in hypercube!
Good communication network!

Modular Arithmetic and multiplicative inverses.

Example:

\( 3^{-1} \pmod{7} \)?
\[ 5 \]
\( 5^{-1} \pmod{7} \)?
\[ 3 \]
Inverse Unique? Yes.
Proof: \( a \) and \( b \) inverses of \( x \pmod{n} \)
\[ ax = bx = 1 \pmod{n} \]
\[ ab = bx = b \pmod{n} \]
\[ a = b \pmod{n} \]
\( 3^{-1} \pmod{6} \)? No, no, no....
\( 3 \pmod{6} \)

See,... no inverse!

...Modular Arithmetic...

Arithmetic modulo \( m \).
Elements of equivalence classes of integers:
\[ \{0, \ldots, m-1\} \]
and integer \( i = a \pmod{m} \)
if \( i = a \pmod{km} \) for integer \( k \).
or if the remainder of \( i \) divided by \( m \) is \( a \).
Can do calculations by taking remainders at the beginning
or at the end.
\[ 58 + 32 = 90 = 6 \pmod{7} \]
\[ 58 + 32 = 2 + 4 = 6 \pmod{7} \]
\[ 58 + 32 = 2 + 3 = -1 = 6 \pmod{7} \]
Negative numbers work the way you are used to.
\[ -3 = 0 - 3 = 7 - 3 = 4 \pmod{7} \]
Additive inverses are intuitively negative numbers.

Modular Arithmetic Inverses and GCD

\( x \) has inverse modulo \( m \) if and only if \( \gcd(x, m) = 1 \).

Group structures more generally.
Proof Idea:
\( \{0, \ldots, (m-1)x\} \) are distinct modulo \( m \) if and only if \( \gcd(x, m) = 1 \).

Finding gcd:
\[ \gcd(x, y) = \gcd(y, x - y) = \gcd(y, x \pmod{y}) \]
Give recursive Algorithm! Base Case? \( \gcd(x, 0) = x \).

Extended gcd\((x, y)\) returns \((d, a, b)\)
\[ d = \gcd(x, y) \] and \[ d = ax + by \]
Multiplicative inverse of \((x, m)\).
\[ \gcd(x, m) = (1, a, b) \]
\( a \) is inverse! \[ 1 = ax + bm = ax \pmod{m} \].

Idea: egcd.
gcd produces 1
by adding and subtracting multiples of \( x \) and \( y \)

Hand calculation: egcd.

Extended GCD: \( \gcd(7, 60) = 1 \).
\[ \gcd(7, 60) \]
\[ 7(0) + 60(1) = 60 \]
\[ 7(1) + 60(0) = 7 \]
\[ 7(0) + 60(1) = 4 \]
\[ 7(-1) + 60(-1) = 3 \]
\[ 7(-1) + 60(2) = 1 \]
Confirm: \(-119 + 120 = 1 \)
\[ d = e^{-1} = -17 = 43 \pmod{60} \]

Midterm format

Time: 120 minutes.
Some short answers.
Get all ideas that you learned.
Know material well: fast, correct.
Know material medium: slower, less correct.
Know material not so well: Uh oh.
Some longer questions.
Proofs, algorithms, properties.
Not so much calculation.
See piazza for more resources.
E.g., TA videos for past exams.
Wrapup.

Other issues....
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Private message on piazza.

Good Studying!!!!!!!!!!!!!!!