Lecture 7. Outline.

1. Quickly finish isoperimetric inequality for hypercube.
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2. Modular Arithmetic.
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   Clock Math!!!
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3. Inverses for Modular Arithmetic: Greatest Common Divisor.
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   Division!!!
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1. Quickly finish isoperimetric inequality for hypercube.
2. Modular Arithmetic.
   Clock Math!!!
3. Inverses for Modular Arithmetic: Greatest Common Divisor.
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4. Euclid’s GCD Algorithm.
1. Quickly finish isoperimetric inequality for hypercube.

2. Modular Arithmetic.
   Clock Math!!!

3. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!

4. Euclid’s GCD Algorithm.
   A little tricky here!
Isoperimetry.

For 3-space:

Surface Area: \(4\pi r^2\), Volume: \(4\frac{1}{3}\pi r^3\).

Ratio: \(\frac{1}{3}r = \Theta\left(\frac{V-1}{3}\right)\).

Graphical Analog: Cut into two pieces and find ratio of edges/vertices on small side.

Tree: \(\Theta\left(\frac{1}{|V|}\right)\).

Hypercube: \(\Theta(1)\).

Surface Area is roughly at least the volume!
Isoperimetry.

For 3-space:

The sphere minimizes surface area to volume.

Surface Area: $4\pi r^2$, Volume: $\frac{4}{3}\pi r^3$.

Ratio: $\frac{1}{3r} = \Theta\left(\frac{V}{V^{\frac{1}{3}}}\right)$.

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Tree: $\Theta\left(\frac{1}{|V|}\right)$.

Hypercube: $\Theta\left(1\right)$.

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Surface Area is roughly at least the volume!
Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.
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An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges ($0x, 1x$).
Recursive Definition.

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An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$.
Thm: Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$;

Terminology: $(S, V - S)$ is cut. $(E \cap S \times (V - S))$ - cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.
Thm: Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$;
Hypercube: Can’t cut me!

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Terminology:
$(S, V - S)$ is cut.
Theorem: Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$; $|E \cap S \times (V - S)| \geq |S|$.

Terminology:
- $(S, V - S)$ is cut.
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**Thm:** Any subset \( S \) of the hypercube where \(|S| \leq |V|/2\) has \( \geq |S|\) edges connecting it to \( V - S \); \(|E \cap S \times (V - S)| \geq |S|\)

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**Restatement:** for any cut in the hypercube, the number of cut edges is at least the size of the small side.
Proof of Large Cuts.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**
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Proof:
Base Case: \(n = 1\)
Proof of Large Cuts.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**
Base Case: \(n = 1\) \(V = \{0,1\}\).
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.
Proof:
Base Case: \(n = 1\) \(V = \{0,1\}\).
\(S = \{0\}\) has one edge leaving.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

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Base Case: \(n = 1\) \(V = \{0,1\}\).
- \(S = \{0\}\) has one edge leaving. \(|S| = \emptyset\) has 0.
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Base Case: \(n = 1\) \(V = \{0, 1\}\).
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Base Case: \(n = 1\) \(V = \{0, 1\}\).
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Induction Step Idea

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.
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Use recursive definition into two subcubes.
Induction Step Idea

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.
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Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

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Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.

Case 2: Count inside and across.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

**Case 1:** Count edges inside subcube inductively.

**Case 2:** Count inside and across.
Induction Step

**Thm:** For any cut \((S, V – S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).
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**Proof:** Induction Step.
Recursive definition:
Induction Step

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.  

**Proof: Induction Step.**  
Recursive definition:  
$H_0 = (V_0, E_0), H_1 = (V_1, E_1)$, edges $E_x$ that connect them.
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step.

Recursive definition:
\[
H_0 = (V_0, E_0), H_1 = (V_1, E_1), \text{ edges } E_x \text{ that connect them.}
\]
\[
H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)
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Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)
Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step.

Recursive definition:

$H_0 = (V_0, E_0), \quad H_1 = (V_1, E_1)$, edges $E_x$ that connect them.

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Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$

Both $S_0$ and $S_1$ are small sides.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step.

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Both \(S_0\) and \(S_1\) are small sides. So by induction.

Edges cut in \(H_0 \geq |S_0|\).
**Induction Step**

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof: Induction Step.**

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**Proof: Induction Step.**

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Total cut edges \(\geq |S_0| + |S_1| = |S|\).
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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Both \(S_0\) and \(S_1\) are small sides. So by induction.
- Edges cut in \(H_0 \geq |S_0|\).
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Total cut edges \(\geq |S_0| + |S_1| = |S|\).
Induction Step. Case 2.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq |V_0|/2. \]
**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$. 
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).


\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)
\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]
\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

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Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[|S_1| \leq |V_1|/2\] since \(|S| \leq |V|/2.\)

\[\implies \geq |S_1|\text{ edges cut in } E_1.\]

\[|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2\]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\(|S_0| \geq |V_0|/2.

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\(|S_1| \leq |V_1|/2\) since \(|S| \leq |V|/2\).

\(\implies \geq |S_1|\) edges cut in \(E_1\).

\(|S_0| \geq |V_0|/2\) \(\implies |V_0 - S| \leq |V_0|/2\)

\(\implies \geq |V_0| - |S_0|\) edges cut in \(E_0\).
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

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|S_0| \geq |V_0|/2.
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Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

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|S_0| \geq |V_0|/2 \Rightarrow |V_0 - S| \leq |V_0|/2
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Edges in \(E_x\) connect corresponding nodes.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]

\[ \Rightarrow \geq |S_1| \text{ edges cut in } E_1. \]

\[ |S_0| \geq |V_0|/2 \Rightarrow |V_0 - S| \leq |V_0|/2 \]

\[ \Rightarrow \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]

Edges in \(E_x\) connect corresponding nodes.

\[ \Rightarrow = |S_0| - |S_1| \text{ edges cut in } E_x. \]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

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Recall Case 1: \(|S_0|, |S_1| \leq |V|/2

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Edges in \(E_x\) connect corresponding nodes.

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Total edges cut:
Induction Step. Case 2.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[|S_0| \geq |V_0|/2.\]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\(|S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2.\)

\[\implies \geq |S_1| \text{ edges cut in } E_1.\]

\[|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2\]

\[\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.\]

Edges in \(E_x\) connect corresponding nodes.

\[\implies = |S_0| - |S_1| \text{ edges cut in } E_x.\]

Total edges cut:

\[\geq\]
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).


\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: \( |S_0|, |S_1| \leq |V|/2 \)
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\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]
\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]
\[ \implies \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]

Edges in \( E_x \) connect corresponding nodes.
\[ \implies = |S_0| - |S_1| \text{ edges cut in } E_x. \]

Total edges cut:
\[ \geq |S_1| \]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq |V_0|/2.\]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2.\]

\[ \implies \geq |S_1| \text{ edges cut in } E_1.\]

\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]

\[ \implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.\]

Edges in \(E_x\) connect corresponding nodes.

\[ \implies = |S_0| - |S_1| \text{ edges cut in } E_x.\]

Total edges cut:

\[ \geq |S_1| + |V_0| - |S_0| \]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step. Case 2.**

\[ |S_0| \geq \frac{|V_0|}{2}. \]

Recall Case 1: \(|S_0|, |S_1| \leq \frac{|V|}{2}\)

\[ |S_1| \leq \frac{|V_1|}{2} \text{ since } |S| \leq \frac{|V|}{2}. \]

\[ \iff \geq |S_1| \text{ edges cut in } E_1. \]

\[ |S_0| \geq \frac{|V_0|}{2} \implies |V_0 - S| \leq \frac{|V_0|}{2} \]

\[ \iff \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]

Edges in \(E_x\) connect corresponding nodes.

\[ \iff = |S_0| - |S_1| \text{ edges cut in } E_x. \]

Total edges cut:

\[ \geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| \]
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).


\(|S_0| \geq |V_0|/2.
\]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2
\]
\(|S_1| \leq |V_1|/2\) since \(|S| \leq |V|/2.
\]
\[\implies |S_1| \text{ edges cut in } E_1.
\]
\[|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2
\]
\[\implies |V_0| - |S_0| \text{ edges cut in } E_0.
\]

Edges in \(E_x\) connect corresponding nodes.
\[\implies = |S_0| - |S_1| \text{ edges cut in } E_x.
\]

Total edges cut:
\[\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|\]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[|S_0| \geq |V_0|/2.\]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[|S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2.\]

\[\implies \geq |S_1| \text{ edges cut in } E_1.\]

\[|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2\]

\[\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.\]

Edges in \(E_x\) connect corresponding nodes.

\[\implies = |S_0| - |S_1| \text{ edges cut in } E_x.\]

Total edges cut:

\[\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|\]

\[|V_0|\]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step. Case 2.**

- \(|S_0| \geq \frac{|V_0|}{2}\).
- Recall Case 1: \(|S_0|, |S_1| \leq \frac{|V|}{2}\)
- \(|S_1| \leq \frac{|V_1|}{2}\) since \(|S| \leq \frac{|V|}{2}\).
- \(\implies \geq |S_1|\) edges cut in \(E_1\).
- \(|S_0| \geq \frac{|V_0|}{2}\) \(\implies |V_0 - S| \leq \frac{|V_0|}{2}\)
- \(\implies \geq |V_0| - |S_0|\) edges cut in \(E_0\).

Edges in \(E_x\) connect corresponding nodes.
- \(\implies = |S_0| - |S_1|\) edges cut in \(E_x\).

Total edges cut:
- \(\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|\)
- \(|V_0| = \frac{|V|}{2} \geq |S|\).
**Induction Step. Case 2.**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[
|S_0| \geq |V_0|/2.
\]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\(|S_1| \leq |V_1|/2\) since \(|S| \leq |V|/2\).

\[\implies \geq |S_1| \text{ edges cut in } E_1.\]

\(|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2\)

\[\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.\]

Edges in \(E_x\) connect corresponding nodes.

\[\implies = |S_0| - |S_1| \text{ edges cut in } E_x.\]

Total edges cut:

\[\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|\]

\[|V_0| = |V|/2 \geq |S|.\]
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).


\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2 \]
\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]
\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]
\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]
\[ \implies \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]

Edges in \(E_x\) connect corresponding nodes. 
\[ \implies = |S_0| - |S_1| \text{ edges cut in } E_x. \]

Total edges cut:
\[ \geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0| \]
\[ |V_0| = |V|/2 \geq |S|. \]

Also, case 3 where \(|S_1| \geq |V|/2\) is symmetric.
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on \( \{0, 1\}^n \).
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0, 1\}^n$.

Central area of study in computer science!
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0, 1\}^n$.

Central area of study in computer science!

Yes/No Computer Programs $\equiv$ Boolean function on $\{0, 1\}^n$
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0,1\}^n$. 

Central area of study in computer science!

Yes/No Computer Programs $\equiv$ Boolean function on $\{0,1\}^n$

Central object of study.
Next Up.

Modular Arithmetic.
Clock Math

If it is 1:00 now.
Clock Math

If it is 1:00 now.
What time is it in 2 hours?

3:00!
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours?

11 / 42
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
What time is it in 15 hours?

11 / 42
If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.
If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
   Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system. Clock time equivalent up to to addition/subtraction of 12.
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
   Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours?
Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
   Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
   Clock time equivalent up to addition/subtraction of 12.

What time is it in 100 hours? 101:00!
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
    Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
  Clock time equivalent up to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
    Actually 4:00.

  16 is the “same as 4” with respect to a 12 hour clock system.
  Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
  \[101 = 12 \times 8 + 5.\]
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
\[ 101 = 12 \times 8 + 5. \]
5 is the same as 101 for a 12 hour clock system.
If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
   Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
   \[101 = 12 \times 8 + 5.\]
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
    Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
  Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
  \[101 = 12 \times 8 + 5.\]
  5 is the same as 101 for a 12 hour clock system.
  Clock time equivalent up to addition of any integer multiple of 12.
If it is 1:00 now.
    What time is it in 2 hours? 3:00!
    What time is it in 5 hours? 6:00!
    What time is it in 15 hours? 16:00!
     Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
   \[ 101 = 12 \times 8 + 5. \]
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in \{12, 1, \ldots, 11\}
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
    Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
    101 = 12 \times 8 + 5.

5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in \{12, 1, \ldots, 11\}
(Almost remainder, except for 12 and 0 are equivalent.)
Day of the week.

Today is Monday.
Day of the week.

Today is Monday.
What day is it a year from now?
Today is Monday.
What day is it a year from now? on February 6, 2018?
Day of the week.

Today is Monday.

What day is it a year from now? on February 6, 2018?
Number days.
Day of the week.

Today is Monday.

What day is it a year from now? on February 6, 2018?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.
Day of the week.

Today is Monday.

What day is it a year from now? on February 6, 2018?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
  5 days from now.

12 / 42
Day of the week.

Today is Monday.

What day is it a year from now? on February 6, 2018?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

5 days from now. day 7
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
  5 days from now. day 7 or day 0
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?

Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
Day of the week.

Today is Monday.

What day is it a year from now? on February 6, 2018?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

5 days from now. day 7 or day 0 or Sunday.

25 days from now.

What day is it a year from now?

This year is not a leap year.

So 365 days from now.

Day 2+365 or day 367.

Smallest representation: subtract 7 until smaller than 7.

divide and get remainder.

367/7 leaves quotient of 52 and remainder 3.

365 = 7(52) + 3

or February 6, 2018 is a Wednesday.
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
  5 days from now. day 7 or day 0 or Sunday.
  25 days from now. day 27

This year is not a leap year.
So 365 days from now.
Day 2+365 or day 367.
Smallest representation:
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6.
Day of the week.

Today is Monday.  
What day is it a year from now?  on February 6, 2018?  
Number days.  
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.  

Today: day 2.  
5 days from now. day 7 or day 0 or Sunday.  
25 days from now. day 27 or day 6. 27 = (7)3 + 6
Day of the week.

Today is Monday.

What day is it a year from now? on February 6, 2018?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

5 days from now. day 7 or day 0 or Sunday.

25 days from now. day 27 or day 6. $27 = (7)3 + 6$

Two days are equivalent up to addition/subtraction of multiple of 7.
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?

Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6. 27 = (7)3 + 6
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6. $27 = (7)3 + 6$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 6
Day of the week.

Today is Monday.

What day is it a year from now? on February 6, 2018?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
  5 days from now. day 7 or day 0 or Sunday.
  25 days from now. day 27 or day 6. \(27 = (7)3 + 6\)
  two days are equivalent up to addition/subtraction of multiple of 7.
  11 days from now is day 6 which is Saturday!
Day of the week.

Today is Monday.
   What day is it a year from now? on February 6, 2018?
   Number days.
       0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
   5 days from now. day 7 or day 0 or Sunday.
   25 days from now. day 27 or day 6. $27 = (7)3 + 6$
       two days are equivalent up to addition/subtraction of multiple of 7.
       11 days from now is day 6 which is Saturday!

What day is it a year from now?
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
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two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 6 which is Saturday!

What day is it a year from now?
This year is not a leap year.
Today is Monday.
What day is it a year from now? on February 6, 2018?
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Today: day 2.
  5 days from now. day 7 or day 0 or Sunday.
  25 days from now. day 27 or day 6. $27 = (7)3 + 6$
    two days are equivalent up to addition/subtraction of multiple of 7.
    11 days from now is day 6 which is Saturday!

What day is it a year from now?
This year is not a leap year. So 365 days from now.
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?

Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6. \(27 = (7)3 + 6\) 
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 6 which is Saturday!

What day is it a year from now?
This year is not a leap year. So 365 days from now.
Day 2+365 or day 367.
Day of the week.

Today is Monday. 
What day is it a year from now? on February 6, 2018? 
Number days. 
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2. 
5 days from now. day 7 or day 0 or Sunday. 
25 days from now. day 27 or day 6. $27 = (7)3 + 6$ 
two days are equivalent up to addition/subtraction of multiple of 7. 
11 days from now is day 6 which is Saturday!

What day is it a year from now? 
This year is not a leap year. So 365 days from now. 
Day 2+365 or day 367. 
Smallest representation:
Today is Monday.
What day is it a year from now? on February 6, 2018?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
  5 days from now. day 7 or day 0 or Sunday.
  25 days from now. day 27 or day 6. \[27 = (7)3 + 6\]
  two days are equivalent up to addition/subtraction of multiple of 7.
  11 days from now is day 6 which is Saturday!

What day is it a year from now?
This year is not a leap year. So 365 days from now.
Day 2+365 or day 367.
Smallest representation:
  subtract 7 until smaller than 7.
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?
Number days.
 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
 5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6. \(27 = (7)3 + 6\)
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 6 which is Saturday!

What day is it a year from now?
This year is not a leap year. So 365 days from now.
Day 2+365 or day 367.
Smallest representation:
  subtract 7 until smaller than 7.
  divide and get remainder.
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6. $27 = (7)3 + 6$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 6 which is Saturday!

What day is it a year from now?
This year is not a leap year. So 365 days from now.
Day 2+365 or day 367.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
367/7
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
  5 days from now. day 7 or day 0 or Sunday.
  25 days from now. day 27 or day 6. \(27 = (7)3 + 6\)
  two days are equivalent up to addition/subtraction of multiple of 7.
  11 days from now is day 6 which is Saturday!

What day is it a year from now?
  This year is not a leap year. So 365 days from now.
  Day 2+365 or day 367.
Smallest representation:
  subtract 7 until smaller than 7.
  divide and get remainder.
  367/7 leaves quotient of 52 and remainder 3.
Day of the week.

Today is Monday.
What day is it a year from now? on February 6, 2018?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6. \(27 = (7)3 + 6\)
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 6 which is Saturday!

What day is it a year from now?
This year is not a leap year. So 365 days from now.
Day 2+365 or day 367.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
367/7 leaves quotient of 52 and remainder 3. \(365 = 7(52) + 3\)
Day of the week.

Today is Monday.

What day is it a year from now? on February 6, 2018?
Number days.
0 for Sunday, 1 for Monday, ... , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6. $27 = (7)3 + 6$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 6 which is Saturday!

What day is it a year from now?
This year is not a leap year. So 365 days from now.
Day 2+365 or day 367.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
367/7 leaves quotient of 52 and remainder 3. $365 = 7(52) + 3$
or February 6, 2018 is a Wednesday.
Day of the week.

Today is Monday.

What day is it a year from now? on February 6, 2018?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6. $27 = (7)3 + 6$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 6 which is Saturday!

What day is it a year from now?
This year is not a leap year. So 365 days from now.
Day 2+365 or day 367.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
367/7 leaves quotient of 52 and remainder 3. $365 = 7(52) + 3$
or February 6, 2018 is a Wednesday.
Years and years...

80 years from now?
Years and years...

80 years from now? 20 leap years.
Years and years...

80 years from now? 20 leap years. \(366 \times 20\) days
Years and years...

80 years from now? 20 leap years. 366 × 20 days
60 regular years.
Years and years...

80 years from now? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Years and years...

80 years from now? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 2.
Years and years...

80 years from now? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. 

Equivalent to?
Hmm.

What is remainder of 366 when dividing by 7?

$52 \times 7 + 2$.

What is remainder of 365 when dividing by 7?

Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60$.

Or February 7, 2096 is Thursday!
Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $2 + 2 \times 6 + 1 \times 4$.

Or Day 4.
February 6, 2095 is Thursday.

"Reduce" at any time in calculation!
Years and years...

80 years from now?  20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 2.

It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

...
Years and years...

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Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

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What is remainder of 366 when dividing by 7?

Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60$.
Remainder when dividing by 7?

Or February 7, 2096 is Thursday!
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February 6, 2095 is Thursday.
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Years and years...

80 years from now? 20 leap years. $366 \times 20$ days
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Hmm.
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What is remainder of 365 when dividing by 7? 1
Years and years...

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   What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
   What is remainder of 365 when dividing by 7? 1
Today is day 2.
   Get Day: $2 + 2 \times 20 + 1 \times 60$
Years and years...

80 years from now? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Years and years...

80 years from now? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7?
Years and years...

80 years from now? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7? $102 = 14 \times 7$
Years and years...

80 years from now? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7? $102 = 14 \times 7 + 4$. 

Or February 7, 2096 is Thursday!
Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$.
Or Day 4.
February 6, 2095 is Thursday.
"Reduce" at any time in calculation!
Years and years...

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What is remainder of 365 when dividing by 7? 1
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Years and years...

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60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
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Further Simplify Calculation:
   20 has remainder 6 when divided by 7.
   60 has remainder 4 when divided by 7.
Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$. 
Years and years...

80 years from now?  20 leap years. 366 × 20 days
   60 regular years. 365 × 60 days
Today is day 2.
It is day 2 + 366 × 20 + 365 × 60. Equivalent to?

Hmm.
   What is remainder of 366 when dividing by 7? 52 × 7 + 2.
   What is remainder of 365 when dividing by 7? 1
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Today is day 2.
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Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$.
  Or Day 4.  February 6, 2095 is Thursday.

“Reduce” at any time in calculation!
Modular Arithmetic: refresher.

\[ x \text{ is congruent to } y \text{ modulo } m \text{ or } \langle x \equiv y \pmod{m} \rangle \]

if and only if \((x - y)\) is divisible by \(m\).
Modular Arithmetic: refresher.

$x$ is congruent to $y$ modulo $m$ or “$x \equiv y \pmod{m}$” if and only if $(x - y)$ is divisible by $m$.
...or $x$ and $y$ have the same remainder w.r.t. $m$. 

Useful Fact: Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$.

or ”$a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$ $\Rightarrow a + b \equiv c + d \pmod{m}$ and $a \cdot b \equiv c \cdot d \pmod{m}$”

Proof: If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$.
If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer $j$.

Therefore, $a + b = c + d + (k + j)m$ and since $k + j$ is integer.

$\Rightarrow a + b \equiv c + d \pmod{m}$.

Can calculate with representative in $\{0, \ldots, m - 1\}$.
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \pmod{m} \)” if and only if \( (x - y) \) is divisible by \( m \).
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Modular Arithmetic: refresher.

$x$ is congruent to $y$ modulo $m$ or “$x \equiv y \ (\text{mod } m)$” if and only if $(x - y)$ is divisible by $m$.
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Mod 7 equivalence classes:
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Mod 7 equivalence classes:
\[ \{ \ldots, -7, 0, 7, 14, \ldots \} \]
Modular Arithmetic: refresher.

$x$ is congruent to $y$ modulo $m$ or “$x \equiv y \ (\text{mod} \ m)$” if and only if $(x - y)$ is divisible by $m$.

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Mod 7 equivalence classes:

$\{\ldots, -7, 0, 7, 14, \ldots\}$  $\{\ldots, -6, 1, 8, 15, \ldots\}$
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Mod 7 equivalence classes:

$\{\ldots, -7, 0, 7, 14, \ldots\}$  $\{\ldots, -6, 1, 8, 15, \ldots\}$ ...

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$. 

Proof:
If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$.

If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer $j$.

Therefore, $a + b = c + d + (k + j)m$ since $k + j$ is integer.

$\Rightarrow a + b \equiv c + d \pmod{m}$.

Can calculate with representative in $\{0, \ldots, m-1\}$. 

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**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
**Modular Arithmetic: refresher.**

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Mod 7 equivalence classes:
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If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).

Therefore,
Modular Arithmetic: refresher.

*x is congruent to y modulo m* or “$x \equiv y \pmod{m}$” if and only if $(x - y)$ is divisible by $m$.
...or $x$ and $y$ have the same remainder w.r.t. $m$.
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Mod 7 equivalence classes:
{\ldots, -7, 0, 7, 14, \ldots} {\ldots, -6, 1, 8, 15, \ldots} ...

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If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer $j$.
Therefore, $a + b = c + d + (k + j)m$.
Modular Arithmetic: refresher.

\[ x \text{ is congruent to } y \text{ modulo } m \] if and only if \( (x - y) \) is divisible by \( m \).

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Mod 7 equivalence classes:
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)

\[ \implies a + b \equiv c + d \pmod{m} \text{ and } a \cdot b \equiv c \cdot d \pmod{m} \]

**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
Therefore, \( a + b = c + d + (k + j)m \) and since \( k + j \) is integer.

\[ \implies a + b \equiv c + d \pmod{m}. \]

Can calculate with representative in \{0, \ldots, m - 1\}. 

Notation

\( x \mod m \) or \( \text{mod} (x, m) \)
Notation

\[ x \ (\text{mod} \ m) \text{ or } \mod (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).
Notation

\[ x \equiv b \pmod{m} \]

- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).
Notation

\[ x \pmod{m} \text{ or } \operatorname{mod}(x, m) \]

- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \operatorname{mod}(x, m) = x - \lfloor \frac{x}{m} \rfloor m \]
Notation

\[ x \equiv (\mod m) \text{ or } \mod (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \mod (x, m) = x - \lfloor \frac{x}{m} \rfloor m \]
\( \lfloor \frac{x}{m} \rfloor \) is quotient.
**Notation**

\[ x \pmod{m} \text{ or } \text{mod} \ (x,m) \]

- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} \ (x,m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \\
\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.}
\]

\[
\text{mod} \ (29,12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12
\]
Notation

\[ x \pmod{m} \text{ or } \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\text{mod} \ (x, m) = x - \lfloor \frac{x}{m} \rfloor m
\]

\( \lfloor \frac{x}{m} \rfloor \) is quotient.

\[
\text{mod} \ (29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12
\]
Notation

\[ x \ (\text{mod} \ m) \text{ or } \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.}\]

\[
\text{mod} \ (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor ) \times 12 = 29 - (2) \times 12 = 4
\]
Notation

$x \mod m$ or $\text{mod} \ (x, m)$
- remainder of $x$ divided by $m$ in $\{0, \ldots, m-1\}$.

$\text{mod} \ (x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\text{mod} \ (29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = 5$
Notation

\[ x \pmod{m} \text{ or } \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[
\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.}
\]

\[
\text{mod} \ (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = \mathbf{5}
\]

Work in this system.
**Notation**

- \( x \) (mod \( m \)) or \( \text{mod} (x, m) \)
  - remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = X = 5
\]

Work in this system.

\( a \equiv b \) (mod \( m \)).
Notation

\[ x \pmod{m} \text{ or } \text{mod} \left(x, m\right) \]

- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} \left(x, m\right) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} \left(29, 12\right) = 29 - \left\lfloor \frac{29}{12} \right\rfloor \times 12 = 29 - (2) \times 12 = 4 = X = 5
\]

Work in this system.

\[ a \equiv b \pmod{m} \text{.} \]

Says two integers \( a \) and \( b \) are equivalent modulo \( m \).
Notation

\( x \pmod{m} \) or \( \text{mod} \ (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod} \ (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = X = 5 \]

Work in this system.
\( a \equiv b \pmod{m} \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus is** \( m \)
Notation

\( x \pmod{m} \) or \( \text{mod} \ (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[ \text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod} \ (29, 12) = 29 - \left\lfloor \frac{29}{12} \right\rfloor \times 12 = 29 - (2) \times 12 = 5 \]

Work in this system.

\( a \equiv b \pmod{m} \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

6 \( \equiv \)
Notation

$x \ (\text{mod} \ m)$ or $\text{mod} \ (x, m)$
- remainder of $x$ divided by $m$ in $\{0, \ldots, m-1\}$.

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[
\text{mod} \ (29, 12) = 29 - \left\lfloor \frac{29}{12} \right\rfloor \times 12 = 29 - (2) \times 12 = x = 5
\]

Work in this system.
\[a \equiv b \ (\text{mod} \ m).\]
Says two integers $a$ and $b$ are equivalent modulo $m$.

**Modulus** is $m$

$6 \equiv 3 + 3$
Notation

$x \mod m$ or $\text{mod}(x, m)$
- remainder of $x$ divided by $m$ in $\{0, \ldots, m-1\}$.

\[
\text{mod}(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

$\left\lfloor \frac{x}{m} \right\rfloor$ is quotient.

\[
\text{mod}(29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2 \times 12) = 5
\]

Work in this system.

$a \equiv b \pmod{m}$.
Says two integers $a$ and $b$ are equivalent modulo $m$.

**Modulus** is $m$

$6 \equiv 3 + 3 \equiv 3 + 10$
Notation

\[ x \pmod{m} \text{ or } \text{mod}(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod}(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \text{mod}(29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 \]

Work in this system.
\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]
Notation

\( x \pmod{m} \) or \( \text{mod} (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod} (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = X = 5 \]

Work in this system.

\( a \equiv b \pmod{m} \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\( 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \).

\( 6 = \)
Notation

\[ x \pmod{m} \text{ or } \text{mod } (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod } (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod } (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 5 \]

Work in this system.
\[ a \equiv b \pmod{m} \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \]
\[ 6 = 3 + 3 \]
Notation

\[ x \pmod{m} \text{ or } \text{mod} (x,m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} (x,m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \text{mod} (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = \not{x} = 5 \]

Work in this system.
\[ a \equiv b \pmod{m} \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]
\[ 6 = 3 + 3 = 3 + 10 \]
Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\mod(29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 5
\]

Work in this system.

\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus is** \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]
\[ 6 = 3 + 3 = 3 + 10 \pmod{7}. \]
Notation

\[ x \pmod{m} \text{ or } \text{mod} (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} (29, 12) = 29 - \left\lfloor \frac{29}{12} \right\rfloor \times 12 = 29 - (2) \times 12 = X = 5
\]

Work in this system.
\[ a \equiv b \pmod{m} \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]

\[ 6 = 3 + 3 = 3 + 10 \pmod{7}. \]

Generally, not \( 6 \pmod{7} = 13 \pmod{7} \).
**Notation**

\[ x \equiv (\text{mod } m) \text{ or } \text{mod } (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod } (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod } (29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = x = 5
\]

Work in this system.

\( a \equiv b \pmod{m} \).

Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\( 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \).

\( 6 = 3 + 3 = 3 + 10 \pmod{7} \).

Generally, not \( 6 \pmod{7} = 13 \pmod{7} \).
But probably won’t take off points,
Notation

\[ x \pmod{m} \text{ or } \text{mod } (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod } (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[
\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.}
\]

\[
\text{mod } (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 4
\]

Work in this system.

\[
a \equiv b \pmod{m}.
\]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[
6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}.
\]

\[
6 = 3 + 3 = 3 + 10 \pmod{7}.
\]

Generally, not \( 6 \pmod{7} = 13 \pmod{7} \).

But probably won’t take off points, still hard for us to read.
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}. \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \);
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

Multiplicative inverse of \( x \) is \( y \) where \( xy = 1 \);
\( 1 \) is multiplicative identity element.
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \(x\) is \(y\) where \(xy = 1\);

**1 is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \); **1 is multiplicative identity element.**

In modular arithmetic, **1 is the multiplicative identity element.**

**Multiplicative inverse of** \( x \mod m \) **is** \( y \) **with** \( xy = 1 \mod m \).
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) is \( y \) where \( xy = 1; \)

**1 is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \mod m \) is \( y \) with \( xy = 1 \mod m \).

For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \mod 7 \).
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1; \) 
**1** **is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \) **mod** \( m \) **is** \( y \) **with** \( xy = 1 \pmod{m}. \)

For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \pmod{7} \).

Can solve \( 4x = 5 \pmod{7} \).
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) is \( y \) where \( xy = 1 \); 1 is multiplicative identity element.

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \mod m \) is \( y \) with \( xy = 1 \mod m \).

For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \mod 7 \).

Can solve \( 4x = 5 \mod 7 \).

\[ 2 \cdot 4x = 2 \cdot 5 \mod 7 \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \);
**1 is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \mod m \) **is** \( y \) **with** \( xy = 1 \mod m \).

For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \mod 7 \).

Can solve \( 4x = 5 \mod 7 \).
\[ 2 \cdot 4x = 2 \cdot 5 \mod 7 \]
\[ 8x = 10 \mod 7 \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \);

**1 is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \mod m \) **is** \( y \) **with** \( xy = 1 \mod m \).

For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \mod 7 \).

Can solve \( 4x = 5 \mod 7 \).

\[ 2 \cdot 4x = 2 \cdot 5 \mod 7 \]
\[ 8x = 10 \mod 7 \]
\[ x = 3 \mod 7 \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \); **1 is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \) **mod** \( m \) **is** \( y \) **with** \( xy = 1 \) (mod \( m \)).

For 4 modulo 7 inverse is 2: \[ 2 \cdot 4 \equiv 8 \equiv 1 \] (mod 7).

Can solve \( 4x = 5 \) (mod 7).

\[ 2 \cdot 4x = 2 \cdot 5 \] (mod 7)
\[ 8x = 10 \] (mod 7)
\[ x = 3 \] (mod 7)

Check!
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1; \)

**1 is multiplicative identity element.**

In modular arithmetic, **1** is the multiplicative identity element.

**Multiplicative inverse of** \( x \) **mod** \( m \) **is** \( y \) **with** \( xy = 1 \) (**mod** \( m \)).

For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \) (**mod** 7).

Can solve \( 4x = 5 \) (**mod** 7).

\[ 2 \cdot 4x = 2 \cdot 5 \] \( \text{(mod 7)} \)

\[ 8x = 10 \] \( \text{(mod 7)} \)

\[ x = 3 \] \( \text{(mod 7)} \)

Check! \( 4(3) = 12 = 5 \) (**mod** 7).
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \); \n**1 is multiplicative identity element.**

In modular arithmetic, \( 1 \) is the multiplicative identity element.

**Multiplicative inverse of** \( x \mod m \) **is** \( y \) **with** \( xy = 1 \pmod{m} \).

For 4 modulo 7 inverse is 2: \[ 2 \cdot 4 \equiv 8 \equiv 1 \pmod{7}. \]

Can solve \( 4x = 5 \pmod{7} \).
\[ x = 3 \pmod{7} \implies \text{Check!} \ 4(3) = 12 = 5 \pmod{7}. \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1; \) **1 is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element. **Multiplicative inverse of** \( x \mod m \) **is** \( y \) **with** \( xy = 1 \mod m).**

For 4 modulo 7 inverse is 2: \[ 2 \cdot 4 \equiv 8 \equiv 1 \mod 7. \]

Can solve \( 4x = 5 \mod 7). \]
\[ x = 3 \mod 7 \implies \text{Check! } 4(3) = 12 = 5 \mod 7. \]

For 8 modulo 12: no multiplicative inverse!
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \(x\) \ is \(y\) where \(xy = 1\);

**1 is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \(x \mod m\) \ is \(y\) with \(xy = 1 \mod m\).

For 4 modulo 7 inverse is 2:

\[ 2 \cdot 4 \equiv 8 \equiv 1 \mod 7. \]

Can solve \(4x = 5 \mod 7\):

\[ x = 3 \mod 7 \implies \text{Check! } 4(3) = 12 = 5 \mod 7. \]

For 8 modulo 12: no multiplicative inverse!

“Common factor of 4”
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies (\frac{1}{2}) \cdot 2x = (\frac{1}{2}) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) is \( y \) where \( xy = 1 \); 1 is multiplicative identity element.

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \) mod \( m \) is \( y \) with \( xy = 1 \) (mod \( m \)).

For 4 modulo 7 inverse is 2: \[ 2 \cdot 4 \equiv 8 \equiv 1 \mod 7. \]

Can solve \( 4x = 5 \) (mod 7). \[ x = 3 \mod 7 \implies \text{Check! } 4(3) = 12 = 5 \mod 7. \]

For 8 modulo 12: no multiplicative inverse!

“Common factor of 4” \( \implies \)
\[ 8k - 12\ell \text{ is a multiple of four for any } \ell \text{ and } k \implies \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \); \( 1 \) **is multiplicative identity element.**

In modular arithmetic, \( 1 \) is the multiplicative identity element.

**Multiplicative inverse of** \( x \) **mod** \( m \) **is** \( y \) **with** \( xy = 1 \) **(mod** \( m \)).

For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \) **(mod** \( 7 \)).

Can solve \( 4x = 5 \) **(mod** \( 7 \)).
\( x = 3 \) **(mod** \( 7 \)) ::: Check! \( 4(3) = 12 = 5 \) **(mod** \( 7 \)).

For 8 modulo 12: no multiplicative inverse!

“Common factor of 4” \( \implies \)
\( 8k - 12\ell \) **is a multiple of four for any** \( \ell \) **and** \( k \) \( \implies \)
\( 8k \not\equiv 1 \) **(mod** \( 12 \)) **for any** \( k \).
Greatest Common Divisor and Inverses.

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If greatest common divisor of $x$ and $m$, $\gcd(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$. 
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**Proof Sketch:**
The set $S = \{0 \cdot x, 1 \cdot x, \ldots, (m-1) \cdot x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$.

For $x = 4$ and $m = 6$. All products of 4... $S = \{0 \cdot 4, 1 \cdot 4, 2 \cdot 4, 3 \cdot 4, 4 \cdot 4, 5 \cdot 4\} = \{0, 4, 8, 12, 16, 20\}$ reducing (mod 6) $S = \{0, 4, 2, 0, 4, 2\}$ Not distinct. Common factor 2. Can't be 1. No inverse.

For $x = 5$ and $m = 6$. $S = \{0 \cdot 5, 1 \cdot 5, 2 \cdot 5, 3 \cdot 5, 4 \cdot 5, 5 \cdot 5\} = \{0, 5, 4, 3, 2, 1\}$ All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

Multiply both sides by 5. $x = 15 = 3 \pmod{6}$

4 $x = 3 \pmod{6}$ No solutions. Can't get an odd.

2 $x = 2, 5 \pmod{6}$ Two solutions!
Proof review. Consequence.

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S = \{0, 4, 2, 0, 4, 2\}
\]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]
All distinct,
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

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For \( x = 4 \) and \( m = 6 \). All products of 4...

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All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).
Thm: If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

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Proof review. Consequence.

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\[ 5x = 3 \mod 6 \]

What is $x$? Multiply both sides by 5.

\[ x = 15 \]
Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

Proof Sketch: The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

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What is \( x \)? Multiply both sides by 5.

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x = 15 = 3 \pmod{6}
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Proof review. Consequence.

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Very different for elements with inverses.
Proof Review 2: Bijections.

If $\gcd(x, m) = 1$. 

$x = 3$, $m = 4$. 

$f(1) = 3(1) \equiv 3 \pmod{4}$, 

$f(2) = 6 \equiv 2 \pmod{4}$, 

$f(3)$ is undefined since $x = 3$. 

Oh yeah.

$f(0) = 0$. 

Not a bijection. 

$x = 2$, $m = 4$. 

$f(1) = 2$, $f(2) = 0$, $f(3) = 2$. 

Oh yeah.

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Bijection $\equiv$ unique pre-image and same size. 

All the images are distinct. 

$\Rightarrow$ unique pre-image for any image.
Proof Review 2: Bijective Functions.

If \( \gcd(x,m) = 1 \).

Then the function \( f(a) = xa \mod m \) is a bijection.

One-to-one: there is a unique pre-image.

Onto: the sizes of the domain and co-domain are the same.

\[ x = 3, \quad m = 4. \]

\[ f(1) = 3(1) = 3 \mod 4, \]
\[ f(2) = 6 = 2 \mod 4, \]
\[ f(3) = 1 \mod 4. \]

Oh yeah.

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\[ x = 2, \quad m = 4. \]

\[ f(1) = 2, \quad f(2) = 0, \quad f(3) = 2. \]

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Not a bijection.
Proof Review 2: Bijections.

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Proof Review 2: Bijections.

If \(\gcd(x,m) = 1\).

Then the function \(f(a) = xa \mod m\) is a bijection.

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\(x = 3, m = 4\).
Proof Review 2: Bijections.

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Proof Review 2: Bijections.

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One to one: there is a unique pre-image.
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Proof Review 2: Bijectons.

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Bijection \( \equiv \) unique pre-image and same size.
If \( \gcd(x, m) = 1 \).

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Bijection \( \equiv \) unique pre-image and same size.

All the images are distinct. \( \implies \) unique pre-image for any image.
If $\gcd(x,m) = 1$.
   Then the function $f(a) = xa \mod m$ is a bijection.
   One to one: there is a unique pre-image.
   Onto: the sizes of the domain and co-domain are the same.

$x = 3, m = 4$.
   $f(1) = 3(1) = 3 \pmod{4}, f(2) = 6 = 2 \pmod{4}, f(3) = 1 \pmod{3}$.
   Oh yeah. $f(0) = 0$.

Bijection $\equiv$ unique pre-image and same size.
   All the images are distinct. $\implies$ unique pre-image for any image.

$x = 2, m = 4$. 
If $\gcd(x,m) = 1$.

Then the function $f(a) = xa \mod m$ is a bijection.

One to one: there is a unique pre-image.

Onto: the sizes of the domain and co-domain are the same.

$x = 3, m = 4$.

$f(1) = 3(1) = 3 \mod 4, f(2) = 6 = 2 \mod 4, f(3) = 1 \mod 3$.

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Onto: the sizes of the domain and co-domain are the same.

$x = 3, \; m = 4$.

$f(1) = 3(1) = 3 \mod 4, \; f(2) = 6 = 2 \mod 4, \; f(3) = 1 \mod 3$.

Oh yeah. $f(0) = 0$.

Bijection $\equiv$ unique pre-image and same size.

All the images are distinct. $\implies$ unique pre-image for any image.

$x = 2, \; m = 4$.

$f(1) = 2, \; f(2) = 0, \; f(3) = 2$.

Oh yeah.
Proof Review 2: Bijectons.

If $\gcd(x,m) = 1$.
   Then the function $f(a) = xa \mod m$ is a bijection.
   One to one: there is a unique pre-image.
   Onto: the sizes of the domain and co-domain are the same.

$x = 3, m = 4$.
   $f(1) = 3(1) = 3 \equiv 3 \pmod{4}, f(2) = 6 = 2 \equiv 2 \pmod{4}, f(3) = 1 \equiv 1 \pmod{3}$.
   Oh yeah. $f(0) = 0$.

Bijection $\equiv$ unique pre-image and same size.
   All the images are distinct. $\implies$ unique pre-image for any image.

$x = 2, m = 4$.
   $f(1) = 2, f(2) = 0, f(3) = 2$
   Oh yeah. $f(0) = 0$. 
If \( \gcd(x,m) = 1 \).

Then the function \( f(a) = xa \mod m \) is a bijection.

One to one: there is a unique pre-image.

Onto: the sizes of the domain and co-domain are the same.

\( x = 3, m = 4 \).

\[ f(1) = 3(1) = 3 \mod 4, f(2) = 6 = 2 \mod 4, f(3) = 1 \mod 3. \]

Oh yeah. \( f(0) = 0 \).

Bijection \( \equiv \) unique pre-image and same size.

All the images are distinct. \( \implies \) unique pre-image for any image.

\( x = 2, m = 4 \).

\[ f(1) = 2, f(2) = 0, f(3) = 2 \]

Oh yeah. \( f(0) = 0 \).

Not a bijection.
Finding inverses.

How to find the inverse?

- Find $\text{gcd}(x, m)$.
- Greater than 1? No multiplicative inverse.
- Equal to 1? Multiplicative inverse.

Algorithm:
- Try all numbers up to $x$ to see if it divides both $x$ and $m$.
- Very slow.
Finding inverses.

How to find the inverse?
How to find if $x$ has an inverse modulo $m$?
Finding inverses.

How to find the inverse?
How to find if $x$ has an inverse modulo $m$?
Find $\text{gcd} \ (x, m)$.
Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find $\gcd (x, m)$.
  Greater than 1?
Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find $\text{gcd} \ (x, m)$.
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How to find the inverse?

How to find if \( x \) has an inverse modulo \( m \)?

Find \( \gcd(x, m) \).

Greater than 1? No multiplicative inverse.

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How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find gcd $(x, m)$.
- Greater than 1? No multiplicative inverse.
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Algorithm: Try all numbers up to $x$ to see if it divides both $x$ and $m$.

Very slow.
Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find $\gcd(x, m)$.
   Greater than 1? No multiplicative inverse.
   Equal to 1? Multiplicative inverse.

Algorithm: Try all numbers up to $x$ to see if it divides both $x$ and $m$.

Very slow.
Inverses

Next up.
Inverses

Next up.
Inverses

Next up.

Euclid’s Algorithm.
Inverses

Next up.

Euclid’s Algorithm.
Runtime.
Inverses

Next up.

Euclid’s Algorithm.
  Runtime.
Euclid’s Extended Algorithm.
Does 2 have an inverse mod 8?

No. Any multiple of 2 is 2 away from 0 + 8k for any k \in \mathbb{N}.

Does 2 have an inverse mod 9?

Yes. 5 \cdot 2 \equiv 10 \equiv 1 \pmod{9}.

Does 6 have an inverse mod 9?

No. Any multiple of 6 is 3 away from 0 + 9k for any k \in \mathbb{N}.

3 = \gcd(6, 9) \neq 1.

x has an inverse modulo m if and only if \gcd(x, m) > 1?

No.

\gcd(x, m) = 1?

Yes. Now what?: Compute \gcd!

Compute Inverse modulo m.
Does 2 have an inverse mod 8? No.
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$. 

Does 2 have an inverse mod 9? Yes.
$5 \cdot 2 \equiv 10 \equiv 1 \mod 9$. 

Does 6 have an inverse mod 9? No.
Any multiple of 6 is 3 away from 0 + 9k for any $k \in \mathbb{N}$. 

$3 = \gcd(6, 9) = 3$. 

$x$ has an inverse modulo $m$ if and only if $\gcd(x, m) = 1$. 

Now what?: Compute $\gcd$!
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from 0 + 8k for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9?
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.
Does 2 have an inverse mod 9? Yes.
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
Does 2 have an inverse mod 8? No.
    Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.
Does 2 have an inverse mod 9? Yes. 5
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
$2(5) = 10 = 1$ mod 9.
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from 0 + 8k for any \( k \in \mathbb{N} \).
Does 2 have an inverse mod 9? Yes. 5
   \( 2(5) = 10 = 1 \mod 9 \).
Does 6 have an inverse mod 9?
Does 2 have an inverse mod 8? No.
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Does 2 have an inverse mod 9? Yes. 5
   $2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
   $2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from 0 + 8k for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
   $2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from 0 + 9k for any $k \in \mathbb{N}$.
   $3 = gcd(6, 9)$!
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from 0 + 8k for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
   $2(5) = 10 = 1 \mod 9$.

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   Any multiple of 6 is 3 away from 0 + 9k for any $k \in \mathbb{N}$.
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$x$ has an inverse modulo $m$ if and only if
Does 2 have an inverse mod 8? No.
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Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
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$x$ has an inverse modulo $m$ if and only if
$\gcd(x, m) > 1$?
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
   \[ 2(5) = 10 = 1 \mod 9. \]

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
   \[ 3 = \gcd(6, 9)! \]

$x$ has an inverse modulo $m$ if and only if
   \[ \gcd(x, m) > 1? \text{ No.} \]
   \[ \gcd(x, m) = 1? \]
Does 2 have an inverse mod 8? No.  Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes.  $5$

$2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.  Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.

$3 = \gcd(6,9)!$

$x$ has an inverse modulo $m$ if and only if

$\gcd(x, m) > 1$? No.

$\gcd(x, m) = 1$? Yes.
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from 0 + 8k for any \( k \in \mathbb{N} \).

Does 2 have an inverse mod 9? Yes. 5
   \( 2(5) = 10 = 1 \mod 9 \).

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from 0 + 9k for any \( k \in \mathbb{N} \).
   \( 3 = \gcd(6, 9) \)!

\( x \) has an inverse modulo \( m \) if and only if
   \( \gcd(x, m) > 1 \)? No.
   \( \gcd(x, m) = 1 \)? Yes.

Now what?:
   Compute \( \gcd \)!
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

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$2(5) = 10 = 1 \mod 9$.

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Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
$3 = \gcd(6, 9)$!

$x$ has an inverse modulo $m$ if and only if 
$\gcd(x, m) > 1$? No.
$\gcd(x, m) = 1$? Yes.

Now what?:
Compute $\gcd$!
Compute Inverse modulo $m$. 
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. $5$
   $2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
   $3 = gcd(6,9)$!

$x$ has an inverse modulo $m$ if and only if
   $gcd(x, m) > 1$? No.
   $gcd(x, m) = 1$? Yes.

Now what?:
   Compute gcd!
   Compute Inverse modulo $m$. 
Divisibility...

**Notation:** $d | x$ means “$d$ divides $x$” or
Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$. 
Notation: $d \mid x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d \mid x$ and $d \mid y$ then $d \mid (x + y)$ and $d \mid (x - y)$. 
Notation: $d| x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d| x$ and $d| y$ then $d|(x + y)$ and $d|(x − y)$.

Is it a fact?
Divisibility...

**Notation:** \( d | x \) means “\( d \) divides \( x \)” or 
\[ x = kd \text{ for some integer } k. \]

**Fact:** If \( d | x \) and \( d | y \) then \( d | (x + y) \) and \( d | (x - y) \).

Is it a fact? Yes?
Notation: $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d | x$ and $d | y$ then $d | (x + y)$ and $d | (x - y)$.

Is it a fact? Yes? No?
Notation: \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

Fact: If \( d \mid x \) and \( d \mid y \) then \( d \mid (x + y) \) and \( d \mid (x - y) \).

Is it a fact? Yes? No?

Proof: \( d \mid x \) and \( d \mid y \) or
Notation: \( d \mid x \) means “\( d \) divides \( x \)” or 
\[ x = kd \]
for some integer \( k \).

Fact: If \( d \mid x \) and \( d \mid y \) then \( d \mid (x + y) \) and \( d \mid (x - y) \).

Is it a fact? Yes? No?

Proof: \( d \mid x \) and \( d \mid y \) or
\[ x = \ell d \] and \( y = kd \)
Divisibility...

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Fact:** If $d|x$ and $d|y$ then $d|(x + y)$ and $d|(x - y)$.

Is it a fact? Yes? No?

**Proof:** $d|x$ and $d|y$ or $x = ℓd$ and $y = kd$

$\implies x - y = kd - ℓd$
Notation: $d | x$ means “$d$ divides $x$” or
$x = kd$ for some integer $k$.

Fact: If $d | x$ and $d | y$ then $d | (x + y)$ and $d | (x - y)$.

Is it a fact? Yes? No?

Proof: $d | x$ and $d | y$ or
$x = \ell d$ and $y = kd$

$\implies x - y = kd - \ell d = (k - \ell)d$
Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d|x$ and $d|y$ then $d|(x + y)$ and $d|(x - y)$.

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$\implies x - y = kd - \ell d = (k - \ell)d \implies d|(x - y)$
Divisibility...

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Fact:** If $d | x$ and $d | y$ then $d | (x + y)$ and $d | (x - y)$.

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$x = \ell d \text{ and } y = kd$

$\implies x - y = kd - \ell d = (k - \ell)d \implies d | (x - y)$
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$. 

**Lemma 1:** If $d | x$ and $d | y$, then $d | y$ and $d | \text{mod}(x, y)$.

**Proof:**
\[
\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y = x - \lfloor s \rfloor \cdot y \text{ for integer } s = kd - s \ell d \text{ for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d = (k - s \ell) d \]

Therefore $d | \text{mod}(x, y)$.

And $d | y$ since it is in condition.

**Lemma 2:** If $d | y$ and $d | \text{mod}(x, y)$, then $d | y$ and $d | x$.

**Proof:** Similar.

**GCD Mod Corollary:** \[\text{gcd}(x, y) = \text{gcd}(y, \text{mod}(x, y))\].

**Proof:** $x$ and $y$ have the same set of common divisors as $x$ and $\text{mod}(x, y)$ by Lemma 1 and 2.

Same common divisors $\Rightarrow$ largest is the same.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$. 

**Lemma 1:**
If $d | x$ and $d | y$, then $d | y$ and $d | \text{mod}(x, y)$.

**Proof:**
$\text{mod}(x, y) = x - \lfloor \frac{x}{y} \rfloor \cdot y = x - \lfloor s \rfloor \cdot y$ for integer $s = kd - \ell$ for integers $k$, $\ell$ where $x = kd$ and $y = \ell d$. Therefore $d | \text{mod}(x, y)$.

And $d | y$ since it is in condition.

**Lemma 2:**
If $d | y$ and $d | \text{mod}(x, y)$, then $d | y$ and $d | x$.

**Proof:**
Similar.

**GCD Mod Corollary:**
$\gcd(x, y) = \gcd(y, \text{mod}(x, y))$.

**Proof:**
$x$ and $y$ have the same set of common divisors as $x$ and $\text{mod}(x, y)$ by Lemma 1 and 2. The same common divisors $\Rightarrow$ largest is the same.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod}(x, y)$.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} (x, y)$.

**Proof:**
\[
\text{mod} (x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y
\]
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} (x, y)$.

**Proof:**
\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]

**Lemma 2:** If $d | y$ and $d | \text{mod} (x, y)$ then $d | y$ and $d | x$.

**Proof:** Similar.

**GCD Mod Corollary:** $\gcd (x, y) = \gcd (y, \text{mod} (x, y))$.

**Proof:** $x$ and $y$ have the same set of common divisors as $x$ and $\text{mod} (x, y)$ by Lemma 1 and 2. Same common divisors $\Rightarrow$ largest is the same.
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\[
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d|\text{mod}(x,y)$.

**Proof:**

\[
\text{mod}(x,y) = x - [x/y] \cdot y \\
= x - [s] \cdot y \quad \text{for integer } s \\
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]
**More divisibility**

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d| \text{mod}(x, y)$.

**Proof:**

\[
\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]

Therefore $d| \text{mod}(x, y)$.
More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d|\text{mod}(x, y)$.

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\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y
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= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
\[
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s\ell)d
\]
Therefore $d | \text{mod}(x, y)$. And $d|y$ since it is in condition.
More divisibility

**Notation:** \( d | x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d | x \) and \( d | y \) then \( d | y \) and \( d | \text{mod} (x, y) \).

**Proof:**

\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]

Therefore \( d | \text{mod} (x, y) \). And \( d | y \) since it is in condition. \( \square \)
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod} (x, y) \).

**Proof:**

\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y
\]

\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]

\[
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]

\[
= (k - s\ell)d
\]

Therefore \( d \mid \text{mod} (x, y) \). And \( d \mid y \) since it is in condition.

**Lemma 2:** If \( d \mid y \) and \( d \mid \text{mod} (x, y) \) then \( d \mid y \) and \( d \mid x \).

**Proof...:** Similar.
More divisibility

**Notation:** \( d | x \) means “\( d \) divides \( x \)” or 
\( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d | x \) and \( d | y \) then \( d | y \) and \( d | \text{mod} (x, y) \).

**Proof:**
\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell) d
\]

Therefore \( d | \text{mod} (x, y) \). And \( d | y \) since it is in condition. \( \Box \)

**Lemma 2:** If \( d | y \) and \( d | \text{mod} (x, y) \) then \( d | y \) and \( d | x \).

**Proof...:** Similar. Try this at home.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} (x, y)$.

**Proof:**

$\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y$

$= x - \lfloor s \rfloor \cdot y$ for integer $s$

$= kd - s\ell d$ for integers $k, \ell$ where $x = kd$ and $y = \ell d$

$= (k - s\ell)d$

Therefore $d | \text{mod} (x, y)$. And $d | y$ since it is in condition. 

**Lemma 2:** If $d | y$ and $d | \text{mod} (x, y)$ then $d | y$ and $d | x$.

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**Notation:** $d|x$ means “$d$ divides $x$” or
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\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
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= (k - s\ell)d
\]
Therefore $d|\text{mod}(x, y)$. And $d|y$ since it is in condition. \qed

**Lemma 2:** If $d|y$ and $d|\text{mod}(x, y)$ then $d|y$ and $d|x$.

**Proof...:** Similar. Try this at home. \ish

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \text{mod}(x, y))$. 

More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} \ (x, y)$.

**Proof:**

\[
\text{mod} \ (x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y
\]

\[
= x - \left\lfloor \frac{s}{y} \right\rfloor \cdot y \quad \text{for integer } s
\]

\[
= kd - s \ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]

\[
= (k - s \ell )d
\]

Therefore $d | \text{mod} \ (x, y)$. And $d | y$ since it is in condition.

**Lemma 2:** If $d | y$ and $d | \text{mod} \ (x, y)$ then $d | y$ and $d | x$.

**Proof...:** Similar. Try this at home.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \ \text{mod} \ (x, y))$.

**Proof:** $x$ and $y$ have **same** set of common divisors as $x$ and $\text{mod} \ (x, y)$ by Lemma 1 and 2.
More divisibility

**Notation:** \(d|x\) means “\(d\) divides \(x\)” or \(x = kd\) for some integer \(k\).

**Lemma 1:** If \(d|x\) and \(d|y\) then \(d|y\) and \(d|\text{mod}(x, y)\).

**Proof:**
\[
\text{mod}(x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]
Therefore \(d|\text{mod}(x, y)\). And \(d|y\) since it is in condition. □

**Lemma 2:** If \(d|y\) and \(d|\text{mod}(x, y)\) then \(d|y\) and \(d|x\).

**Proof...:** Similar. Try this at home. □ish.

**GCD Mod Corollary:** \(\gcd(x, y) = \gcd(y, \text{mod}(x, y))\).

**Proof:** \(x\) and \(y\) have **same** set of common divisors as \(x\) and \(\text{mod}(x, y)\) by Lemma 1 and 2.

Same common divisors \(\implies\) largest is the same.
More divisibility

**Notation:** \(d|\) \(x\) means “\(d\) divides \(x\)” or 
\[x = kd\] for some integer \(k\).

**Lemma 1:** If \(d|\) \(x\) and \(d|\) \(y\) then \(d|\) \(y\) and \(d|\) \(\text{mod}(x, y)\).

**Proof:**
\[
\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]
Therefore \(d|\) \(\text{mod}(x, y)\). And \(d|\) \(y\) since it is in condition.

**Lemma 2:** If \(d|\) \(y\) and \(d|\) \(\text{mod}(x, y)\) then \(d|\) \(y\) and \(d|\) \(x\).

**Proof...:** Similar. Try this at home.

**GCD Mod Corollary:** \(\gcd(x, y) = \gcd(y, \text{mod}(x, y))\).

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Same common divisors \(\implies\) largest is the same.
Euclid’s algorithm.

**GCD Mod Corollary**: \( \gcd(x, y) = \gcd(y, \ \text{mod} \ (x, y)) \).
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)?
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? 7
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)?

7 since 7 divides 7 and 7 divides 0.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \ \text{mod} \ (x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)?
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod{x, y}) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \mod (x, y)) \).

Hey, what’s \( \text{gcd}(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \text{gcd}(x, 0) \)? x

(define (euclid x y)
   (if (= y 0)
       x
       (euclid y (mod x y)))) ***
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)?  \(  x \)

\[
\text{(define (euclid x y)}
\text{ (if (= y 0)}
\text{ x)
\text{ (euclid y (mod x y)))}) \]

***

**Theorem:** \( \text{(euclid x y)} = \gcd(x, y) \) if \( x \geq y \).
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \mod(x, y)) \).

Hey, what’s \( \text{gcd}(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \text{gcd}(x, 0) \)? \( x \)

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))) ***
```

**Theorem:** \( (\text{euclid } x \ y) = \text{gcd}(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x,y) = \text{gcd}(y, \mod(x,y)) \).

Hey, what's \( \text{gcd}(7,0) \)? \( 7 \) since 7 divides 7 and 7 divides 0

What's \( \text{gcd}(x,0) \)? \( x \)

\[
(\text{define (euclid } x \ y) \\
(\text{if } (= \ y \ 0) \\
\ x \\
(\text{euclid } \ y \ (\mod \ x \ y)))) \\
***
\]

**Theorem:** \( \text{(euclid } x \ y) = \text{gcd}(x,y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”. 
Euclid’s algorithm.

GCD Mod Corollary: \( \text{gcd}(x, y) = \text{gcd}(y, \text{mod}(x, y)) \).

Hey, what’s \( \text{gcd}(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \text{gcd}(x, 0) \)? \( x \)

```
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))) ***
```

**Theorem:** \( (\text{euclid } x \ y) = \text{gcd}(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.
**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”
  \( \implies \) “\( x \) is common divisor and clearly largest.”
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

\[
\text{(define (euclid x y)}
\text{ (if (= y 0)}
\text{ x)
\text{ (euclid y (mod x y))))) ***}
\]

**Theorem:** \( (\text{euclid } x y) = \gcd(x, y) \) if \( x \geq y \).

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**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”
\[
\implies \text{“} x \text{ is common divisor and clearly largest.”}
\]

**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

\[
\text{(define (euclid x y)}
\text{  (if (= y 0)}
\text{    x)}
\text{  (euclid y (mod x y)))}}) \quad ***
\]

**Theorem:** \((\text{euclid } x y) = \gcd(x, y)\) if \( x \geq y \).

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call in line (***)) meets conditions plus arguments “smaller”
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

```
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))
```  

**Theorem:** \( (\text{euclid } x \ y) = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0, \) “\( x \) divides \( y \) and \( x \)”

\( \implies \) “\( x \) is common divisor and clearly largest.”

**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)

call in line (***)) meets conditions plus arguments “smaller”
and by strong induction hypothesis
Euclid’s algorithm.

**GCD Mod Corollary:** \(\gcd(x, y) = \gcd(y, \mod(x, y))\).

Hey, what’s \(\gcd(7, 0)\)? 7 since 7 divides 7 and 7 divides 0

What’s \(\gcd(x, 0)\)?  \(x\)

```
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))) ***
```

**Theorem:** \((\text{euclid } x y) = \gcd(x, y)\) if \(x \geq y\).

**Proof:** Use Strong Induction.

**Base Case:** \(y = 0\), “\(x\) divides \(y\) and \(x\)”

\(\implies \text{ “}x\text{ is common divisor and clearly largest.”}\)

**Induction Step:** \(\mod(x, y) < y \leq x\) when \(x \geq y\)

call in line (***)) meets conditions plus arguments “smaller”
and by strong induction hypothesis
computes \(\gcd(y, \mod(x, y))\)
Euclid’s algorithm.

**GCD Mod Corollary:** \[ \text{gcd}(x, y) = \text{gcd}(y, \mod(x, y)). \]

Hey, what’s \( \text{gcd}(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \text{gcd}(x, 0) \)? \( x \)

```
(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y))))
```  

**Theorem:** \( (\text{euclid } x \ y) = \text{gcd}(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”

\[ \implies \text{“} x \text{ is common divisor and clearly largest.”} \]

**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)

call in line (***)) meets conditions plus arguments “smaller”
and by strong induction hypothesis
computes \( \text{gcd}(y, \mod(x, y)) \)
which is \( \text{gcd}(x, y) \) by GCD Mod Corollary.
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? $7$ since $7$ divides $7$ and $7$ divides $0$
What’s $\gcd(x, 0)$? $x$

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

***

**Theorem:** $(\text{euclid} \ x \ y) = \gcd(x, y)$ if $x \geq y$.

Proof: Use Strong Induction.
Base Case: $y = 0$, “$x$ divides $y$ and $x$”
           $\implies$ “$x$ is common divisor and clearly largest.”

Induction Step: mod $(x, y) < y \leq x$ when $x \geq y$

call in line (*** ) meets conditions plus arguments “smaller”
and by strong induction hypothesis
computes $\gcd(y, \mod(x, y))$
which is $\gcd(x, y)$ by GCD Mod Corollary.
Excursion: Value and Size.

Before discussing running time of gcd procedure...
Excursion: Value and Size.

Before discussing running time of gcd procedure...

What is the value of 1,000,000?
Excursion: Value and Size.

Before discussing running time of gcd procedure...
What is the value of 1,000,000?
one million or 1,000,000!
Excursion: Value and Size.

Before discussing running time of gcd procedure...
What is the value of 1,000,000?
one million or 1,000,000!
What is the “size” of 1,000,000?
Excursion: Value and Size.

Before discussing running time of gcd procedure...
What is the value of $1,000,000$?
one million or $1,000,000$!
What is the “size” of $1,000,000$?
Number of digits in base 10: 7.
Before discussing running time of gcd procedure...

What is the value of 1,000,000?

one million or 1,000,000!

What is the “size” of 1,000,000?

Number of digits in base 10: 7.

Number of bits (a digit in base 2): 21.
Before discussing running time of gcd procedure...
What is the value of 1,000,000?
one million or 1,000,000!
What is the “size” of 1,000,000?
Number of digits in base 10: 7.
Number of bits (a digit in base 2): 21.
For a number $x$, what is its size in bits?
Excursion: Value and Size.

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What is the value of 1,000,000? 
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Number of digits in base 10: 7.
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For a number $x$, what is its size in bits?

$$n = b(x) \approx \log_2 x$$
Excursion: Value and Size.

Before discussing running time of gcd procedure...

What is the value of 1,000,000?
One million or 1,000,000!

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Number of digits in base 10: 7.
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For a number $x$, what is its size in bits?

$$n = b(x) \approx \log_2 x$$
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) “divisions” where \(n = b(x) \approx \log_2 x\).
Euclid procedure is fast.

**Theorem:** $(\text{euclid } x \ y)$ uses $2n$ "divisions" where $n = b(x) \approx \log_2 x$.

Is this good?
Euclid procedure is fast.

**Theorem:** \( (\text{euclid } x \ y) \) uses \( 2n \) ”divisions” where \( n = b(x) \approx \log_2 x \).

Is this good? Better than trying all numbers in \( \{2, \ldots, y/2\} \)?
Euclid procedure is fast.

**Theorem:** (euclid \( x, y \)) uses \( 2n \) ”divisions” where \( n = b(x) \approx \log_2 x \). Is this good? Better than trying all numbers in \( \{2, \ldots, y/2\} \)?

Check 2,
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) ”divisions” where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots, y/2\}\)?

Check 2, check 3,
Euclid procedure is fast.

**Theorem:** (euclid x y) uses $2n$ ”divisions” where $n = b(x) \approx \log_2 x$. Is this good? Better than trying all numbers in $\{2, \ldots, y/2\}$?

Check 2, check 3, check 4,
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) ”divisions” where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots, y/2\}\)?

Check 2, check 3, check 4, check 5 . . . , check \(y/2\).
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) ”divisions” where \(n = b(x) \approx \log_2 x\).

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Check 2, check 3, check 4, check 5 . . . , check \(y/2\).
Euclid procedure is fast.

**Theorem:** (euclid x y) uses $2n$ ”divisions” where $n = b(x) \approx \log_2 x$.

Is this good? Better than trying all numbers in $\{2, \ldots, y/2\}$?

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

If $y \approx x$
Euclid procedure is fast.

**Theorem:** $(\text{euclid } x \ y)$ uses $2n$ ”divisions” where $n = b(x) \approx \log_2 x$.

Is this good? Better than trying all numbers in $\{2, \ldots \ y/2\}$?

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

If $y \approx x$ roughly $y$ uses $n$ bits
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) "divisions" where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots y/2\}\)?

Check 2, check 3, check 4, check 5 \ldots, check \(y/2\).

If \(y \approx x\) roughly \(y\) uses \(n\) bits …

\(2^{n-1}\) divisions! Exponential dependence on size!
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) "divisions" where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots \ y/2\}\)?

Check 2, check 3, check 4, check 5 \ldots, check \(y/2\).

If \(y \approx x\) roughly \(y\) uses \(n\) bits ...

\(2^{n-1}\) divisions! Exponential dependence on size!

101 bit number.
Euclid procedure is fast.

**Theorem:** (euclid x y) uses $2n$ ”divisions” where $n = b(x) \approx \log_2 x$.

Is this good? Better than trying all numbers in \{2, \ldots y/2\}? Check 2, check 3, check 4, check 5 . . . , check $y/2$.

If $y \approx x$ roughly $y$ uses $n$ bits ...

$2^{n-1}$ divisions! Exponential dependence on size!

101 bit number. $2^{100} \approx 10^{30} =$ “million, trillion, trillion” divisions!
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x y)\) uses \(2n\) "divisions" where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots y/2\}\)?

Check 2, check 3, check 4, check 5 . . . , check \(y/2\).

If \(y \approx x\) roughly \(y\) uses \(n\) bits . . .

\(2^{n-1}\) divisions! Exponential dependence on size!

101 bit number. \(2^{100} \approx 10^{30} = \text{“million, trillion, trillion” divisions!}\)

\(2n\) is much faster!
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) ”divisions” where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots y/2\}\)?

Check 2, check 3, check 4, check 5 . . . , check \(y/2\).

If \(y \approx x\) roughly \(y\) uses \(n\) bits ...  
\(2^{n-1}\) divisions! Exponential dependence on size!

101 bit number. \(2^{100} \approx 10^{30} = \) “million, trillion, trillion” divisions!

\(2n\) is much faster! .. roughly 200 divisions.
Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

"(gcd x y)"

euclid(700, 568)
euclid(568, 132)
euclid(132, 40)
euclid(40, 12)
euclid(12, 4)
euclid(4, 0)

4

Notice: The first argument decreases rapidly. At least a factor of 2 in two recursive calls. (The second is less than the first.)
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$. 

Notice: The first argument decreases rapidly. At least a factor of 2 in two recursive calls. (The second is less than the first.) 

\[
euclid(700, 568) \\
euclid(568, 132) \\
euclid(132, 40) \\
euclid(40, 12) \\
euclid(12, 4) \\
euclid(4, 0) \\
4
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( y/2 \).
“(gcd x y)” at work.

\[
euclid(700, 568)
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( y/2 \).
“(gcd x y)” at work.

\[
euclid(700, 568) \\
euclid(568, 132)
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . ., check $y/2$.
“(gcd x y)” at work.

\[
\begin{align*}
\text{euclid}(700,568) \\
\text{euclid}(568,132) \\
\text{euclid}(132,40)
\end{align*}
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.

euclid(700, 568)
euclid(568, 132)
euclid(132, 40)
euclid(40, 12)
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check y/2.
“(gcd x y)” at work.

    euclid(700, 568)
    euclid(568, 132)
    euclid(132, 40)
    euclid(40, 12)
    euclid(12, 4)
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( y/2 \).
“(gcd \( x \ y \))” at work.

\[
\begin{align*}
\text{euclid}(700, 568) \\
\text{euclid}(568, 132) \\
\text{euclid}(132, 40) \\
\text{euclid}(40, 12) \\
\text{euclid}(12, 4) \\
\text{euclid}(4, 0)
\end{align*}
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.

\begin{verbatim}
euclid(700, 568)
   euclid(568, 132)
      euclid(132, 40)
         euclid(40, 12)
            euclid(12, 4)
               euclid(4, 0)
                  4
\end{verbatim}
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( y/2 \).
“(gcd x y)” at work.

\[
\begin{align*}
euclid(700, 568) \\
euclid(568, 132) \\
euclid(132, 40) \\
euclid(40, 12) \\
euclid(12, 4) \\
euclid(4, 0) \\
4
\end{align*}
\]

Notice: The first argument decreases rapidly.
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . ., check y/2.
“(gcd x y)” at work.

```
euclid(700, 568)  
  euclid(568, 132)  
    euclid(132, 40)  
      euclid(40, 12)  
        euclid(12, 4)  
          euclid(4, 0)  
            4
```

Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( y/2 \).
“(gcd x y)” at work.

\[
\begin{align*}
\text{euclid}(700, 568) \\
\text{euclid}(568, 132) \\
\text{euclid}(132, 40) \\
\text{euclid}(40, 12) \\
\text{euclid}(12, 4) \\
\text{euclid}(4, 0)
\end{align*}
\]

Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.
(The second is less than the first.)
Maybe Break.
Runtime Proof.

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$. 
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ "divisions" where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.
After $2 \log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.
Runtime Proof.

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Theorem: (euclid x y) uses \(O(n)\) "divisions" where \(n = b(x)\).

Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.

After \(2\log_2 x = O(n)\) recursive calls, argument \(x\) is 1 bit number.
One more recursive call to finish.
Runtime Proof.

```
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))
```

**Theorem:** (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

After $2 \log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number. One more recursive call to finish.
1 division per recursive call.
Runtime Proof.

```
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))
```

**Theorem:** (euclid x y) uses $O(n)$ "divisions" where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

After $2 \log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.
One more recursive call to finish.
1 division per recursive call.
$O(n)$ divisions.
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.
(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < \frac{x}{2} \), first argument is \( y \)
\[ \implies \text{true in one recursive call}; \]
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\implies$ true in one recursive call;
Runtime Proof (continued.)

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))
```

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

**Proof of Fact:** Recall that first argument decreases every call.

Case 1: \( y < x/2 \), first argument is \( y \)
\[ \implies \text{true in one recursive call;} \]

Case 2: Will show \( y \geq x/2 \) \( \implies \text{“} \mod(x, y) \leq x/2. \text{“} \)
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < \frac{x}{2} \), first argument is \( y \)
  \[ \Rightarrow \text{true in one recursive call;} \]

Case 2: Will show \( y \geq \frac{x}{2} \) \( \Rightarrow \) \( \text{mod}(x, y) \leq \frac{x}{2} \). 
mod \( (x, y) \) is second argument in next recursive call,
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\implies$ true in one recursive call;

Case 2: Will show “$y \geq x/2$” $\implies$ “$\text{mod}(x,y) \leq x/2$.”

  $\text{mod}(x,y)$ is second argument in next recursive call, and becomes the first argument in the next one.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\implies$ true in one recursive call;

Case 2: Will show "$y \geq x/2$" $\implies$ "$\text{mod}(x, y) \leq x/2$".
  $\text{mod} (x, y)$ is second argument in next recursive call,
  and becomes the first argument in the next one.
When $y \geq x/2$, then
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < x/2 \), first argument is \( y \)
      \( \implies \) true in one recursive call;

Case 2: Will show “\( y \geq x/2 \) \( \implies \) “\( mod(x, y) \leq x/2 \).”

\( mod(x, y) \) is second argument in next recursive call,
and becomes the first argument in the next one.

When \( y \geq x/2 \), then

\[ \left\lfloor \frac{x}{y} \right\rfloor = 1, \]
(define (euclid x y)
 (if (= y 0)
     x
     (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\implies$ true in one recursive call;

Case 2: Will show “$y \geq x/2$” $\implies$ “$\text{mod}(x, y) \leq x/2.$”

  $\text{mod}(x, y)$ is second argument in next recursive call, and becomes the first argument in the next one.

When $y \geq x/2$, then

  $\left\lfloor \frac{x}{y} \right\rfloor = 1$,

  $\text{mod}(x, y) = x - y\left\lfloor \frac{x}{y} \right\rfloor =$
Runtime Proof (continued.)

```
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))
```

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < x/2 \), first argument is \( y \)
  \( \implies \) true in one recursive call;

Case 2: Will show \( y \geq x/2 \) \( \implies \) “\( \text{mod}(x, y) \leq x/2 \)”

\( \text{mod} (x, y) \) is second argument in next recursive call,
and becomes the first argument in the next one.

When \( y \geq x/2 \), then

\[
\left\lfloor \frac{x}{y} \right\rfloor = 1,
\]

\[
\text{mod} (x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = x - y \leq x - x/2
\]
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\implies$ true in one recursive call;

Case 2: Will show “$y \geq x/2$” $\implies$ “$\text{mod}(x, y) \leq x/2$.”

  $\text{mod} \ (x, y)$ is second argument in next recursive call, and becomes the first argument in the next one.

When $y \geq x/2$, then

  $\lfloor \frac{x}{y} \rfloor = 1,$

  $\text{mod} \ (x, y) = x - y \lfloor \frac{x}{y} \rfloor = x - y \leq x - x/2 = x/2$
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < \frac{x}{2}$, first argument is $y$
  $\implies$ true in one recursive call;

Case 2: Will show $“y \geq \frac{x}{2}” \implies “\text{mod}(x, y) \leq \frac{x}{2}.”$

  mod ($x, y$) is second argument in next recursive call, and becomes the first argument in the next one.

When $y \geq \frac{x}{2}$, then

\[
\left\lfloor \frac{x}{y} \right\rfloor = 1,
\]

\[
\text{mod} (x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = x - y \leq x - \frac{x}{2} = \frac{x}{2}
\]
Finding an inverse?

We showed how to efficiently tell if there is an inverse.
Finding an inverse?

We showed how to efficiently tell if there is an inverse.
Extend euclid to find inverse.
Euclid’s GCD algorithm.

(define (euclid x y)
 (if (= y 0)
     x
     (euclid y (mod x y))))
Euclid’s GCD algorithm.

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Computes the gcd(x, y) in $O(n)$ divisions.
Euclid’s GCD algorithm.

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Computes the gcd($x, y$) in $O(n)$ divisions.
For $x$ and $m$, if gcd($x, m$) = 1 then $x$ has an inverse modulo $m$. 
Multiplicative Inverse.

GCD algorithm used to tell if there is a multiplicative inverse.
Multiplicative Inverse.

GCD algorithm used to tell if there is a multiplicative inverse. How do we find a multiplicative inverse?
Euclid’s Extended GCD Theorem:
For any $x, y$ there are integers $a, b$ where

$ax + by = \gcd(x, y)$. 

"Make $d$ out of sum of multiples of $x$ and $y$."

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$ax + bm = 1$

$ax \equiv 1 \pmod{m}$.

So $a$ multiplicative inverse of $x$ ($\mod m$)!!

Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$(3) \cdot 12 + (-1) \cdot 35 = 1$. 

$a = 3$ and $b = -1$.

The multiplicative inverse of 12 ($\mod 35$) is 3.
Euclid’s Extended GCD Theorem:
For any \( x, y \) there are integers \( a, b \) where

\[ ax + by \]
Extended GCD

Euclid’s Extended GCD Theorem:
For any $x, y$ there are integers $a, b$ where

$$ax + by = d \quad \text{where } d = \gcd(x, y).$$
Extended GCD

Euclid’s Extended GCD Theorem:
For any $x, y$ there are integers $a, b$ where

$$ax + by = d \quad \text{where } d = \text{gcd}(x, y).$$

“Make $d$ out of sum of multiples of $x$ and $y$.”

Example: For $x = 12$ and $y = 35$, $\text{gcd}(12, 35) = 1$.

$$3 \cdot 12 + (-1) \cdot 35 = 1.$$ 

$a = 3$ and $b = -1$.

The multiplicative inverse of 12 (mod 35) is 3.
Extended GCD

Euclid’s Extended GCD Theorem:
For any $x, y$ there are integers $a, b$ where

$$ax + by = d$$

where $d = \gcd(x, y)$.

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?
Euclid’s Extended GCD Theorem:
For any \( x, y \) there are integers \( a, b \) where
\[
ax + by = d \quad \text{where } d = \gcd(x, y).
\]

“Make \( d \) out of sum of multiples of \( x \) and \( y \).”

What is multiplicative inverse of \( x \) modulo \( m \)?

By extended GCD theorem, when \( \gcd(x, m) = 1 \).
Euclid’s Extended GCD Theorem:
For any $x, y$ there are integers $a, b$ where
\[ ax + by = d \quad \text{where} \quad d = \gcd(x, y). \]

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.
\[ ax + bm = 1 \]
Euclid’s Extended GCD Theorem:
For any $x, y$ there are integers $a, b$ where
\[ ax + by = d \quad \text{where } d = \gcd(x, y). \]

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.
\[
ax + bm = 1 \\
ax \equiv 1 - bm \equiv 1 \pmod{m}.
\]
Extended GCD

Euclid’s Extended GCD Theorem:
For any $x, y$ there are integers $a, b$ where
$$ax + by = d \quad \text{where } d = \gcd(x, y).$$

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?
By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$ 

So $a$ multiplicative inverse of $x \pmod{m}$!!
Euclid’s Extended GCD Theorem:
For any \( x, y \) there are integers \( a, b \) where
\[
ax + by = d \quad \text{where} \quad d = \gcd(x, y).
\]

“Make \( d \) out of sum of multiples of \( x \) and \( y \).”

What is multiplicative inverse of \( x \) modulo \( m \)?

By extended GCD theorem, when \( \gcd(x, m) = 1 \).
\[
ax + bm = 1
\]
\[
ax \equiv 1 \pmod{m} \equiv 1 \quad (\text{mod} \quad m).
\]

So \( a \) multiplicative inverse of \( x \) \((\text{mod} \quad m)\)!!

Example: For \( x = 12 \) and \( y = 35 \), \( \gcd(12, 35) = 1 \).
Extended GCD

Euclid’s Extended GCD Theorem:
For any \( x, y \) there are integers \( a, b \) where
\[
ax + by = d \quad \text{where } d = \gcd(x, y).
\]

“Make \( d \) out of sum of multiples of \( x \) and \( y \).”

What is multiplicative inverse of \( x \) modulo \( m \)?

By extended GCD theorem, when \( \gcd(x, m) = 1 \).
\[
ax + bm = 1 \\
ax \equiv 1 - bm \equiv 1 \pmod{m}.
\]

So \( a \) multiplicative inverse of \( x \) \( \pmod{m} \)!!

Example: For \( x = 12 \) and \( y = 35 \), \( \gcd(12, 35) = 1 \).
\[
(3)12 + (-1)35 = 1.
\]
Extended GCD

Euclid’s Extended GCD Theorem:
For any \(x, y\) there are integers \(a, b\) where
\[
ax + by = d \quad \text{where } d = \gcd(x, y).
\]

“Make \(d\) out of sum of multiples of \(x\) and \(y\).”

What is multiplicative inverse of \(x\) modulo \(m\)?

By extended GCD theorem, when \(\gcd(x, m) = 1\).
\[
ax + bm = 1
\]
\[
ax \equiv 1 - bm \equiv 1 \pmod{m}.
\]

So \(a\) multiplicative inverse of \(x\) \((\mod m)!!

Example: For \(x = 12\) and \(y = 35\), \(\gcd(12, 35) = 1\).
\[
(3)12 + (-1)35 = 1.
\]

\(a = 3\) and \(b = -1\).
Extended GCD

Euclid’s Extended GCD Theorem:
For any \(x, y\) there are integers \(a, b\) where

\[
ax + by = d \quad \text{where } d = \gcd(x, y).
\]

“Make \(d\) out of sum of multiples of \(x\) and \(y\).”

What is multiplicative inverse of \(x\) modulo \(m\)?

By extended GCD theorem, when \(\gcd(x, m) = 1\).

\[
ax + bm = 1
\]

\[
ax \equiv 1 - bm \equiv 1 \pmod{m}.
\]

So \(a\) multiplicative inverse of \(x\) \((\mod m)!!\)

Example: For \(x = 12\) and \(y = 35\), \(\gcd(12, 35) = 1\).

\[
(3)12 + (-1)35 = 1.
\]

\(a = 3\) and \(b = -1\).

The multiplicative inverse of 12 \((\mod 35)\) is 3.
Make $d$ out of $x$ and $y$..?

gcd(35, 12)

12 - \lfloor \frac{35}{12} \rfloor \times 12 = 35 - 2 \times 12 = 11

12 - \lfloor \frac{12}{11} \rfloor \times 11 = 12 - 1 \times 11 = 1

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

1 = 12 - (1) \times 11

Get 11 from 35 and 12 and plugin....

Simplify.

$a = 3$ and $b = -1$. 

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Make \( d \) out of \( x \) and \( y \)...

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) ;;& gcd(12, 35 \div 12)
\end{align*}
\]
Make $d$ out of $x$ and $y$...?

$$\gcd(35, 12)$$
$$\gcd(12, 11) ;; \gcd(12, 35 \% 12)$$
$$\gcd(11, 1) ;; \gcd(11, 12 \% 11)$$
Make $d$ out of $x$ and $y$..?

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) &\quad ;; \quad gcd(12, 35 \% 12) \\
gcd(11, 1) &\quad ;; \quad gcd(11, 12 \% 11) \\
gcd(1, 0) &\quad 1
\end{align*}
\]
Make $d$ out of $x$ and $y$..?

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35 \% 12) \\
gcd(11, 1) ;; gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did gcd get 11 from 35 and 12?
Make $d$ out of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) ;; \ gcd(12, 35 \% 12) \\
gcd(11, 1) ;; \ gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]
Make $d$ out of $x$ and $y$..?

\[
\text{gcd}(35, 12) \\
\text{gcd}(12, 11) \;; \; \text{gcd}(12, 35 \mod 12) \\
\text{gcd}(11, 1) \;; \; \text{gcd}(11, 12 \mod 11) \\
\text{gcd}(1, 0) \\
1
\]

How did \text{gcd} get 11 from 35 and 12?

\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \text{gcd} get 1 from 12 and 11?

Get 1 from 12 and 11.

\[
1 = 12 - \left(\frac{1}{11}\right)11 = 12 - \left(\frac{1}{11}\right)\left(35 - \left(\frac{2}{12}\right)12\right) = \left(\frac{3}{12}\right)12 + \left(\frac{-1}{11}\right)35
\]
Make \( d \) out of \( x \) and \( y \)...

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35 \mod 12) \\
gcd(11, 1) ;; gcd(11, 12 \mod 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did \( gcd \) get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \( gcd \) get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]
Make $d$ out of $x$ and $y$..?

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) ;; \ gcd(12, 35 \text{mod} 12) \\
gcd(11, 1) ;; \ gcd(11, 12 \text{mod} 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did gcd get 11 from 35 and 12?
\[35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11\]

How does gcd get 1 from 12 and 11?
\[12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1\]

Algorithm finally returns 1.
Make $d$ out of $x$ and $y$..?

\[
gcd(35,12) \\
gcd(12, 11) ;; \gcd(12, 35 \mod 12) \\
gcd(11, 1) ;; \gcd(11, 12 \mod 11) \\
gcd(1,0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11\]

How does gcd get 1 from 12 and 11?
\[12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?
Make \( d \) out of \( x \) and \( y \)...

\[
gcd(35, 12) \\
gcd(12, 11) \quad ;; \quad gcd(12, 35 \mod 12) \\
gcd(11, 1) \quad ;; \quad gcd(11, 12 \mod 11) \\
gcd(1, 0) \\
1
\]

How did \( gcd \) get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \( gcd \) get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?
Get 1 from 12 and 11.
Make \( d \) out of \( x \) and \( y \) ..?

\[ \begin{align*}
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35 \text{ mod } 12) \\
gcd(11, 1) ;; gcd(11, 12 \text{ mod } 11) \\
gcd(1, 0) \\
1
\end{align*} \]

How did \( \text{gcd} \) get 11 from 35 and 12?

\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \( \text{gcd} \) get 1 from 12 and 11?

\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

\[ 1 = 12 - (1)11 \]
Make \( d \) out of \( x \) and \( y \)...

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) ;; \gcd(12, 35 \mod 12) \\
gcd(11, 1) ;; \gcd(11, 12 \mod 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did \( \gcd \) get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \( \gcd \) get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12)
\]

Get 11 from 35 and 12 and plugin....
Make $d$ out of $x$ and $y$..?

\[
\text{gcd}(35, 12) \\
\text{gcd}(12, 11) ;; \text{gcd}(12, \; 35 \% 12) \\
\text{gcd}(11, 1) ;; \text{gcd}(11, 12 \% 11) \\
\text{gcd}(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[35 - \left\lfloor \frac{35}{12} \right\rfloor \times 12 = 35 - (2) \times 12 = 11\]

How does gcd get 1 from 12 and 11?
\[12 - \left\lfloor \frac{12}{11} \right\rfloor \times 11 = 12 - (1) \times 11 = 1\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[1 = 12 - (1) \times 11 = 12 - (1)(35 - (2) \times 12) = (3)12 + (-1)35\]

Get 11 from 35 and 12 and plugin.... Simplify.
Make $d$ out of $x$ and $y$...

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) & ;; \; gcd(12, 35 \% 12) \\
gcd(11, 1) & ;; \; gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did $gcd$ get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does $gcd$ get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35
\]

Get 11 from 35 and 12 and plugin.... Simplify.
Make $d$ out of $x$ and $y$..?

$$\gcd(35, 12)$$
$$\gcd(12, 11) \ ; ; \ gcd(12, 35 \% 12)$$
$$\gcd(11, 1) \ ; ; \ gcd(11, 12 \% 11)$$
$$\gcd(1, 0)$$
$$1$$

How did gcd get 11 from 35 and 12?
$$35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11$$

How does gcd get 1 from 12 and 11?
$$12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1$$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$$

Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$. 

\[ 36 / 42 \]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y) \\
\quad \text{if } y = 0 \text{ then return }(x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\quad \quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]
\[
\text{if } y = 0 \text{ then return } (x, 1, 0)
\]
\[
\text{else }
\]
\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y))
\]
\[
\text{return } (d, b, a - \text{floor}(x/y) \ast b)
\]

Claim: Returns \((d, a, b)\): \(d = \text{gcd}(a, b)\) and \(d = ax + by\).
Extended GCD Algorithm.

ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x, y))
        return (d, b, a - floor(x/y) * b)

Claim: Returns (d, a, b): \( d = \gcd(a, b) \) and \( d = ax + by \).
Example:

ext-gcd(35, 12)
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]
\[
\text{if } y = 0 \text{ then return }(x, 1, 0)
\]
\[
\text{else}
\]
\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y))
\]
\[
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).
Example:

\[
\text{ext-gcd}(35, 12)
\]
\[
\text{ext-gcd}(12, 11)
\]
Extended GCD Algorithm.

\[ \text{ext-gcd}(x, y) \]
\[
\begin{array}{l}
\text{if } y = 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
\quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\end{array}
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[ \text{ext-gcd}(35, 12) \]
\[ \text{ext-gcd}(12, 11) \]
\[ \text{ext-gcd}(11, 1) \]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]
\[
\begin{align*}
&\text{if } y = 0 \text{ then return } (x, 1, 0) \\
&\text{else} \\
&\quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
&\quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\end{align*}
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[
\begin{align*}
\text{ext-gcd}(35, 12) \\
\text{ext-gcd}(12, 11) \\
\text{ext-gcd}(11, 1) \\
\text{ext-gcd}(1, 0)
\end{align*}
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y) \\
\quad \text{if } y = 0 \text{ then return } (x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\quad \quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).
Example: \(a - \lfloor x/y \rfloor \cdot b = \)

\[
\text{ext-gcd}(35, 12) \\
\quad \text{ext-gcd}(12, 11) \\
\quad \quad \text{ext-gcd}(11, 1) \\
\quad \quad \quad \text{ext-gcd}(1, 0) \\
\quad \quad \text{return } (1, 1, 0) ;; 1 = (1)1 + (0)0
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x,y) \\
\text{if } y = 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
\quad (d, a, b) := \text{ext-gcd}(y, \mod(x,y)) \\
\quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a,b)\) and \(d = ax + by\).

Example: \(a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 11/1 \rfloor \cdot 0 = 1\)

\[
\begin{align*}
\text{ext-gcd}(35, 12) \\
\text{ext-gcd}(12, 11) \\
\quad \text{ext-gcd}(11, 1) \\
\quad \text{ext-gcd}(1, 0) \\
\quad \text{return } (1,1,0) ;; 1 = (1)1 + (0)0 \\
\text{return } (1,0,1) ;; 1 = (0)11 + (1)1
\end{align*}
\]
Extended GCD Algorithm.

ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x, y))
        return (d, b, a - floor(x/y) * b)

Claim: Returns (d, a, b): d = gcd(a, b) and d = ax + by.
Example: a − \lfloor x/y \rfloor \cdot b = 0 − \lfloor 12/11 \rfloor \cdot 1 = −1

ext-gcd(35, 12)
    ext-gcd(12, 11)
    ext-gcd(11, 1)
    ext-gcd(1, 0)
    return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1) ;; 1 = (0)11 + (1)1
    return (1,1,-1) ;; 1 = (1)12 + (-1)11
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]

\[
\text{if } y = 0 \text{ then return}(x, 1, 0)
\]

\[
\text{else}
\]

\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y))
\]

\[
\text{return } (d, b, a - \text{floor}(x/y) \ast b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example: \(a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 35/12 \rfloor \cdot (-1) = 3\)

\[
\text{ext-gcd}(35, 12)
\]

\[
\text{ext-gcd}(12, 11)
\]

\[
\text{ext-gcd}(11, 1)
\]

\[
\text{ext-gcd}(1, 0)
\]

\[
\text{return } (1, 1, 0) ;; 1 = (1)1 + (0)0
\]

\[
\text{return } (1, 0, 1) ;; 1 = (0)11 + (1)1
\]

\[
\text{return } (1, 1, -1) ;; 1 = (1)12 + (-1)11
\]

\[
\text{return } (1, -1, 3) ;; 1 = (-1)35 + (3)12
\]
Extended GCD Algorithm.

ext-gcd(x,y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x,y))
        return (d, b, a - floor(x/y) * b)

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).
Example:

ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
        return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1) ;; 1 = (0)11 + (1)1
      return (1,1,-1) ;; 1 = (1)12 + (-1)11
    return (1,-1, 3) ;; 1 = (-1)35 + (3)12
Theorem:

Returns \((d, a, b)\), where 

\[d = \gcd(a, b)\]  
and  

\[d = ax + by.\]
Extended GCD Algorithm.

```python
ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x, y))
        return (d, b, a - floor(x/y) * b)
```

**Theorem:** Returns \((d, a, b)\), where \(d = \gcd(a, b)\) and

\[ d = ax + by. \]
Correctness.

**Proof:** Strong Induction.$^1$

---

$^1$Assume $d$ is $gcd(x, y)$ by previous proof.
Correctness.

**Proof:** Strong Induction.¹
**Base:** `ext-gcd(x, 0)` returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

---

¹Assume \(d\) is \(\text{gcd}(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.\footnote{Assume $d$ is $gcd(x, y)$ by previous proof.}

Base: $\text{ext-gcd}(x, 0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

Induction Step: Returns $(d, A, B)$ with $d = Ax + By$

Ind hyp: $\text{ext-gcd}(y, \text{mod}(x, y))$ returns $(d, a, b)$ with

\[d = ay + b(\text{mod}(x, y))\]
Correctness.

**Proof:** Strong Induction.\(^1\)

**Base:** \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: \(\text{ext-gcd}(y, \text{mod}(x, y))\) returns \((d, a, b)\) with

\[
d = ay + b(\text{mod}(x, y))
\]

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \text{mod}(x, y))\) so

\(^1\)Assume \(d\) is \(\text{gcd}(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.\(^1\)

Base: \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

Induction Step: Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: \(\text{ext-gcd}(y, \mod(x, y))\) returns \((d, a, b)\) with

\[
d = ay + b(\mod(x, y))
\]

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \mod(x, y))\) so

\[
d = ay + b \cdot (\mod(x, y))
\]

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.¹
Base: ext-gcd(x, 0) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

Induction Step: Returns \((d, A, B)\) with \(d = Ax + By\)
Ind hyp: ext-gcd(y, \(\text{mod}(x, y)\)) returns \((d, a, b)\) with

\[ d = ay + b(\mod(x, y)) \]

ext-gcd(x, y) calls ext-gcd(y, \(\text{mod}(x, y)\)) so

\[
\begin{align*}
    d & = ay + b \cdot (\mod(x, y)) \\
      & = ay + b \cdot (x - \left\lfloor \frac{x}{y} \right\rfloor y)
\end{align*}
\]

¹Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof:** Strong Induction.\(^1\)

**Base:** \texttt{ext-gcd}(x, 0) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: \texttt{ext-gcd}(y, \mod{x, y}) returns \((d, a, b)\) with \(d = ay + b(\mod{x, y})\)

\texttt{ext-gcd}(x, y) calls \texttt{ext-gcd}(y, \mod{x, y}) so

\[
d = ay + b \cdot (\mod{x, y}) \]

\[
= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \]

\[
= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y \]

\(^1\)Assume \(d \) is \(gcd(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.\(^1\)

Base: \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

Induction Step: Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: \(\text{ext-gcd}(y, \text{mod}(x, y))\) returns \((d, a, b)\) with \(d = ay + b(\text{mod}(x, y))\)

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \text{mod}(x, y))\) so

\[
\begin{align*}
    d &= ay + b \cdot (\text{mod}(x, y)) \\
    &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \\
    &= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y
\end{align*}
\]

And \(\text{ext-gcd}\) returns \((d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))\) so theorem holds!

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.\(^1\)

Base: $\text{ext-gcd}(x, 0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

Induction Step: Returns $(d, A, B)$ with $d = Ax + By$

Ind hyp: $\text{ext-gcd}(y, \mod (x, y))$ returns $(d, a, b)$ with $d = ay + b(\mod (x, y))$

$\text{ext-gcd}(x, y)$ calls $\text{ext-gcd}(y, \mod (x, y))$ so

\[
    d = ay + b \cdot (\mod (x, y))
\]

\[
    = ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y)
\]

\[
    = bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y
\]

And $\text{ext-gcd}$ returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds!  

---

\(^1\)Assume $d$ is $\text{gcd}(x, y)$ by previous proof.

Prove: returns \((d, A, B)\) where \(d = Ax + By\).

def ext-gcd(x, y):
    if y == 0:
        return (x, 1, 0)
    else:
        (d, a, b) = ext-gcd(y, mod(x, y))
        return (d, b, a - floor(x/y) * b)

Prove: returns \((d, A, B)\) where \(d = Ax + By\).

\[
\text{ext-gcd}(x,y) \\
\quad \text{if } y = 0 \text{ then return } (x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y)) \\
\quad \quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Recursively: \(d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y)\)
Prove: returns \((d, A, B)\) where \(d = Ax + By\).

\[
\text{ext-gcd}(x, y) \\
\text{if } y = 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
\quad (d, a, b) := \text{ext-gcd}(y, \mod(x, y)) \\
\quad \text{return } (d, b, a \cdot \floor{x/y} \cdot \bmod{y})
\]

Recursively: \(d = ay + b(x - \floor{x/y} \cdot y) \implies d = bx - (a - \floor{x/y}b)y\)

Prove: returns \((d, A, B)\) where \(d = Ax + By\).

\[
\text{ext-gcd}(x, y) \\
\text{if } y = 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Recursively: \(d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y\)

Returns \((d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))\).
Hand Calculation Method for Inverses.

Example: $\gcd(7, 60) = 1$. 

\[ \text{egcd}(7, 60). \]
\[ 7(0) + 60(1) = 60 \]
\[ 7(1) + 60(0) = 7 \]
\[ 7(-8) + 60(1) = 4 \]
\[ 7(9) + 60(-1) = 3 \]
\[ 7(-17) + 60(2) = 1 \]

Confirm: $-119 + 120 = 1$. 

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Hand Calculation Method for Inverses.

Example: $\gcd(7,60) = 1$.

$\text{egcd}(7,60)$. 

Confirm:

$-119 + 120 = 1$ 

$41 / 42$
Hand Calculation Method for Inverses.

Example: \(\gcd(7, 60) = 1\).
\[\text{egcd}(7, 60).\]

\[
7(0) + 60(1) = 60
\]
Example: $\text{gcd}(7, 60) = 1$.

$\text{egcd}(7, 60)$.

\[
7(0) + 60(1) = 60
\]
\[
7(1) + 60(0) = 7
\]
Hand Calculation Method for Inverses.

Example: \( \text{gcd}(7, 60) = 1 \).
\( \text{egcd}(7, 60) \).

\[
\begin{align*}
7(0) + 60(1) &= 60 \\
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\end{align*}
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Hand Calculation Method for Inverses.

Example: $\gcd(7, 60) = 1$.
$\text{egcd}(7, 60)$.

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Example: $\text{gcd}(7, 60) = 1$. $\text{egcd}(7, 60)$.

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Example: \( \gcd(7, 60) = 1 \).
\( \text{egcd}(7, 60) \).

\[
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Hand Calculation Method for Inverses.

Example: $\gcd(7, 60) = 1$.

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\]

Confirm:
Hand Calculation Method for Inverses.

Example: $\gcd(7, 60) = 1$.
$\text{egcd}(7,60)$.

\[
\begin{align*}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7 \\
7(-8) + 60(1) &= 4 \\
7(9) + 60(-1) &= 3 \\
7(-17) + 60(2) &= 1
\end{align*}
\]

Confirm: $-119 + 120 = 1$
Conclusion: Can find multiplicative inverses in $O(n)$ time!
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Very different from elementary school: try 1, try 2, try 3...
Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

Very different from elementary school: try 1, try 2, try 3...

$2^{n/2}$
Conclusion: Can find multiplicative inverses in $O(n)$ time!

Very different from elementary school: try 1, try 2, try 3...

$2^{n/2}$

Inverse of 500,000,357 modulo 1,000,000,000,000,000?
Conclusion: Can find multiplicative inverses in $O(n)$ time!

Very different from elementary school: try 1, try 2, try 3...

$2^{n/2}$

Inverse of 500,000,357 modulo 1,000,000,000,000?

$\leq 80$ divisions.
Conclusion: Can find multiplicative inverses in $O(n)$ time!

Very different from elementary school: try 1, try 2, try 3...

$2^{n/2}$

Inverse of 500,000,357 modulo 1,000,000,000,000?

$\leq$ 80 divisions.

versus 1,000,000
Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!
Very different from elementary school: try 1, try 2, try 3...

$2^{n/2}$

Inverse of 500,000,357 modulo 1,000,000,000,000,000?
$\leq 80$ divisions.
versus 1,000,000
Conclusion: Can find multiplicative inverses in $O(n)$ time!

Very different from elementary school: try 1, try 2, try 3...

$2^{n/2}$

Inverse of 500,000,357 modulo 1,000,000,000,000?

≤ 80 divisions.

versus 1,000,000

Internet Security.
Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!
Very different from elementary school: try 1, try 2, try 3...
$2^{n/2}$

Inverse of 500,000,357 modulo 1,000,000,000,000? 
≤ 80 divisions.
versus 1,000,000

Internet Security.
Public Key Cryptography: 512 digits.
Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

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Inverse of 500,000,357 modulo 1,000,000,000,000,000?

$\leq 80$ divisions.

versus 1,000,000

Internet Security.

Public Key Cryptography: 512 digits.

512 divisions vs.
Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

Very different from elementary school: try 1, try 2, try 3...

$2^{n/2}$

Inverse of 500,000,357 modulo 1,000,000,000,000?

$\leq 80$ divisions.

versus 1,000,000

Internet Security.
Public Key Cryptography: 512 digits.

512 divisions vs.

$(100000000000000000000000000000000000000000)^5$ divisions.
Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!
Very different from elementary school: try 1, try 2, try 3...

$2^{n/2}$

Inverse of 500,000,357 modulo 1,000,000,000,000?

$\leq 80$ divisions.
versus 1,000,000

Internet Security.
Public Key Cryptography: 512 digits.

512 divisions vs.

$(10000000000000000000000000000000000000000000000000)$

5 divisions.

Internet Security:
Conclusion: Can find multiplicative inverses in $O(n)$ time!

Very different from elementary school: try 1, try 2, try 3...

$2^{n/2}$

Inverse of $500,000,357$ modulo $1,000,000,000,000$?

$\leq 80$ divisions.

versus $1,000,000$

Internet Security.
Public Key Cryptography: 512 digits.

512 divisions vs.

$(100000000000000000000000000000000000000000000000000)5$ divisions.

Internet Security: Thursday.