Lecture 6.

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Finish Euler's Formula.

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Planar Five Color theorem.

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Types of graphs.

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Finish Euler's Formula.

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Types of graphs.

- Complete Graphs.
- Trees.
- Hypercubes.

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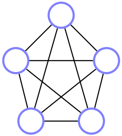
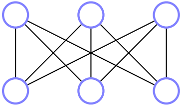
Planarity and Euler





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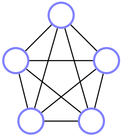

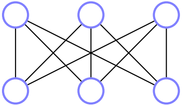

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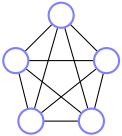

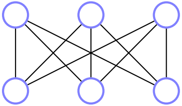

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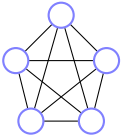

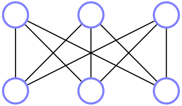

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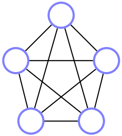

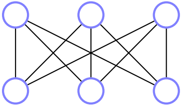

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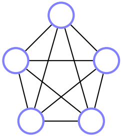

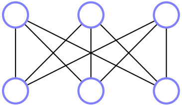

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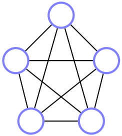

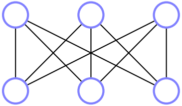

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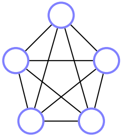

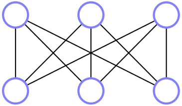

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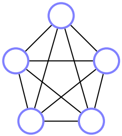

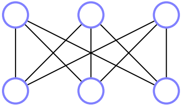

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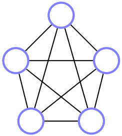

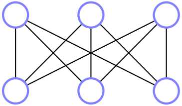

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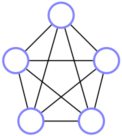

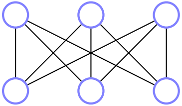

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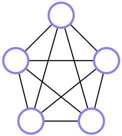

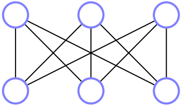

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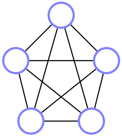

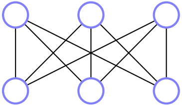

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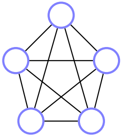
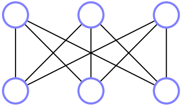
Planarity and Euler





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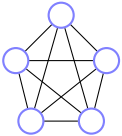

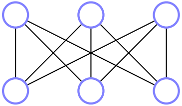

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So......so ...Cool!

Tree.

A tree is a connected acyclic graph.

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To tree or not to tree!

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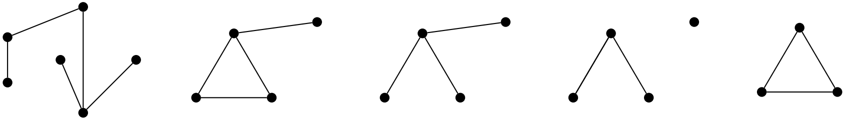

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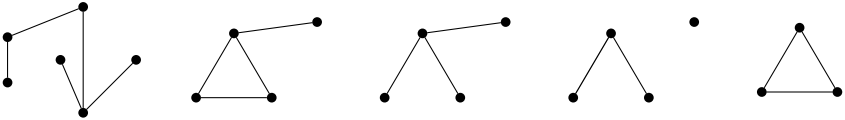

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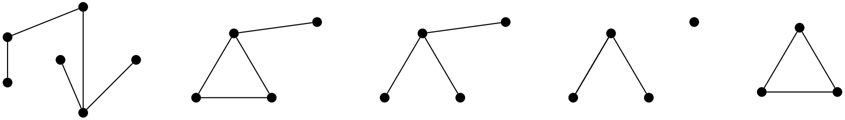

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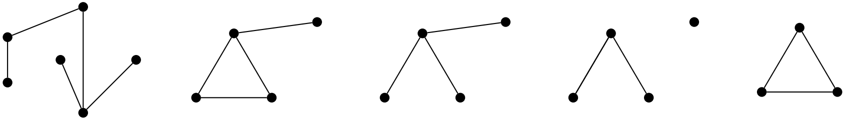

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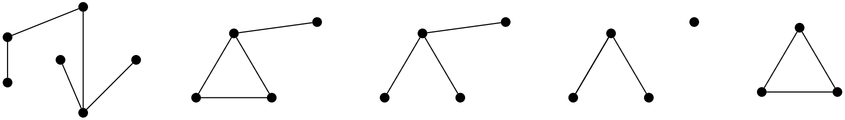

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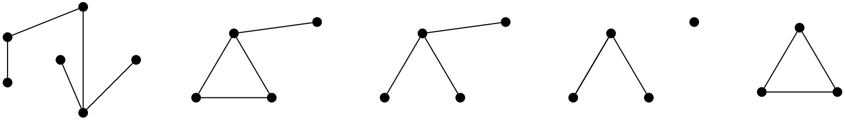

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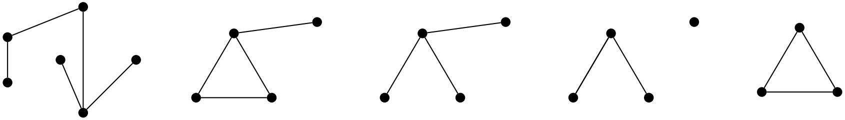

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Faces?

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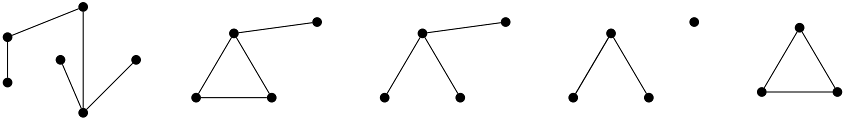

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Faces? 1.

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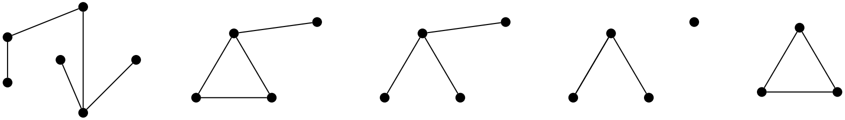

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Faces? 1. 2.

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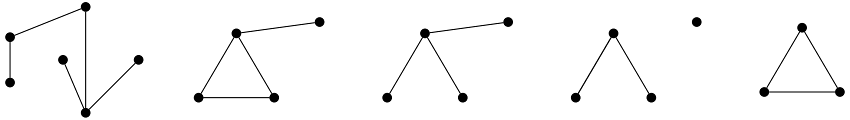

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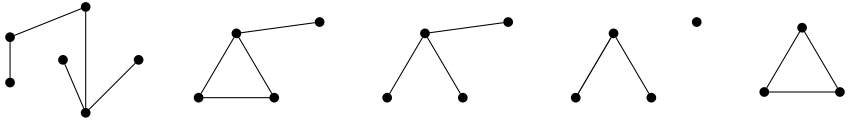

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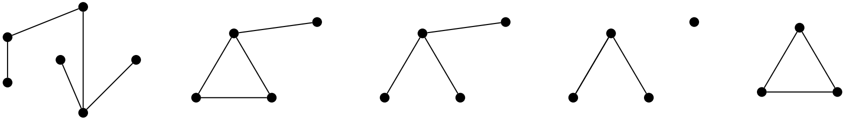

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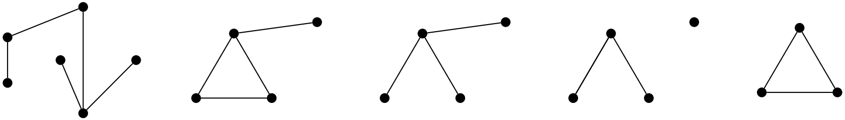

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Faces? 1. 2. 1. 1. 2.
Vertices/Edges.

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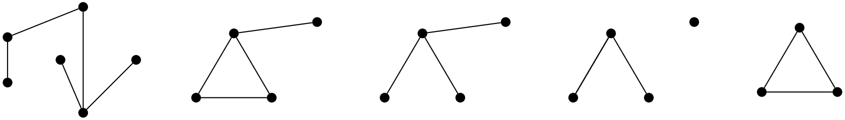

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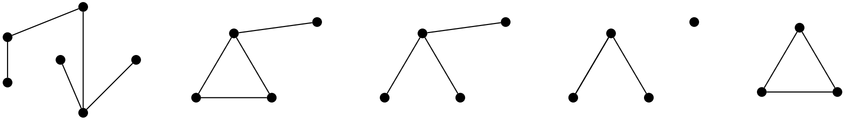

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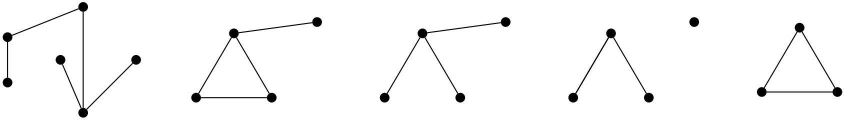

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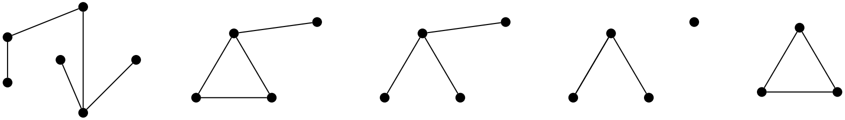

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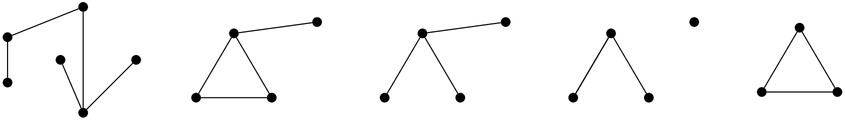

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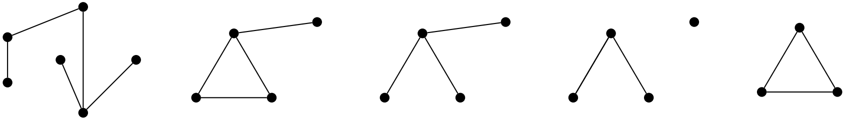

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Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

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Proof:

Euler's formula.

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Proof: Induction on e.

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Base:

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Base: $e = 0$,

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Induction Step:

Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:
If it is a tree.

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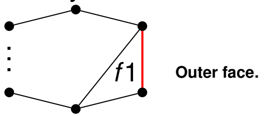

Joins two faces.

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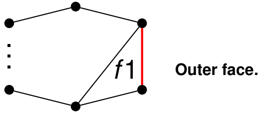

Joins two faces.
New graph: v-vertices.

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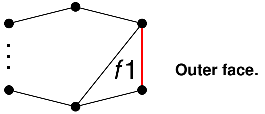

Joins two faces.
New graph: v-vertices. $e-1$ edges.

Euler's formula.

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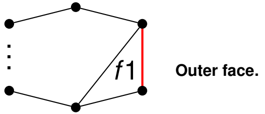

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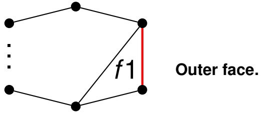

Joins two faces.
New graph: v-vertices. $e - 1$ edges. $f - 1$ faces. Planar.

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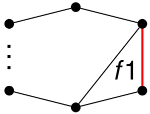

Outer face.

Joins two faces.
New graph: v-vertices. $e-1$ edges. $f-1$ faces. Planar.
$v + (f-1) = (e-1) + 2$ by induction hypothesis.

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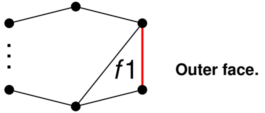

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New graph: v-vertices. $e-1$ edges. $f-1$ faces. Planar.
$v+(f-1)=(e-1)+2$ by induction hypothesis.
Therefore $v+f=e+2$.

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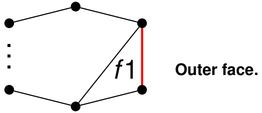

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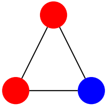

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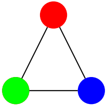

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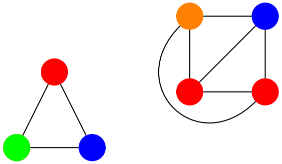

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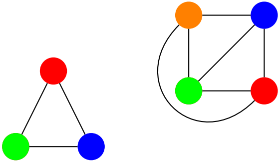

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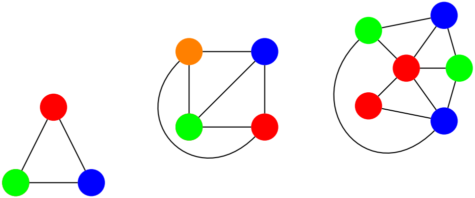

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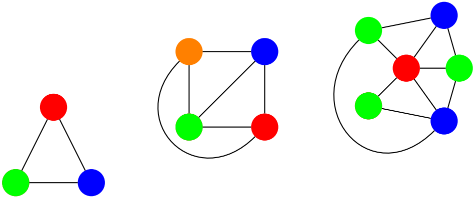

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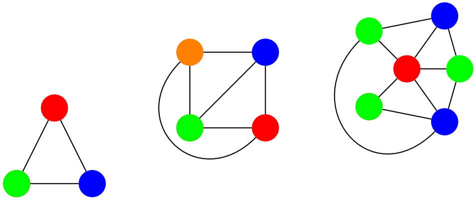

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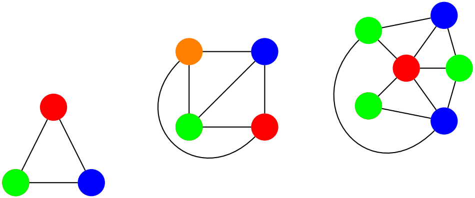

Notice that the last one, has one three colors.

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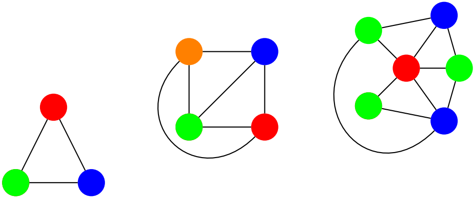

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Fewer colors than number of vertices.

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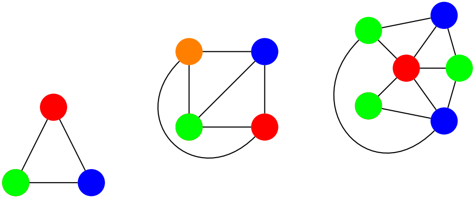

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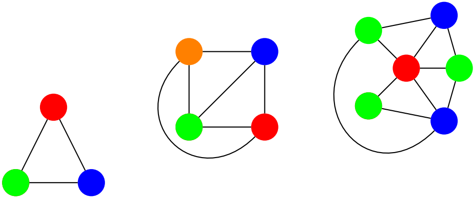

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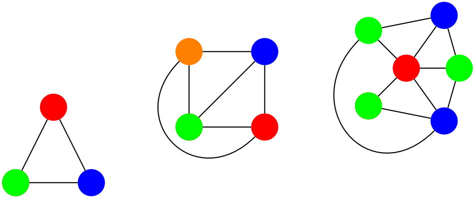

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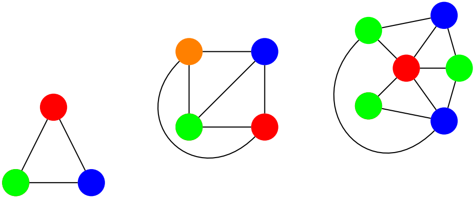

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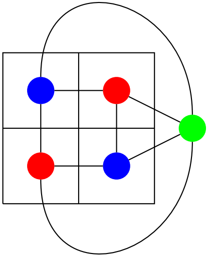
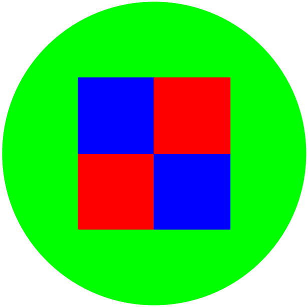
Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





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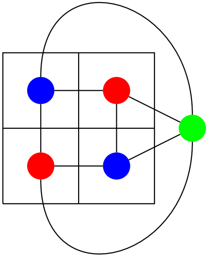

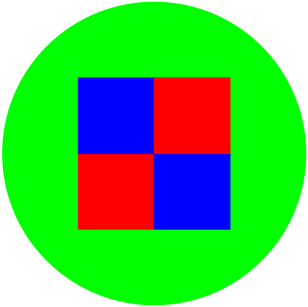

Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

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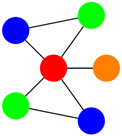
Five color theorem: prelimnary.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.



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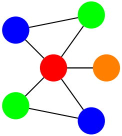

Look at only green and blue.

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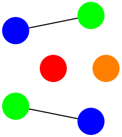

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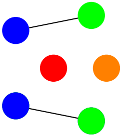

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Connected components.
Can switch in one component.

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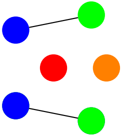

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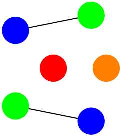

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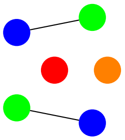

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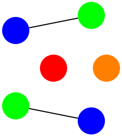

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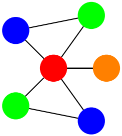

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Proof: Again with the degree 5 vertex.

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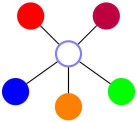
Assume neighbors are colored all differently.



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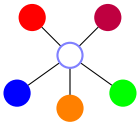
Otherwise one of 5 colors is available.



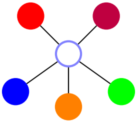
Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.

Assume neighbors are colored all differently.
Otherwise one of 5 colors is available. $\implies$ Done!



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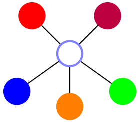

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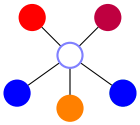

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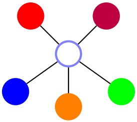

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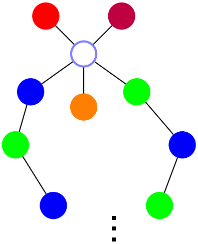

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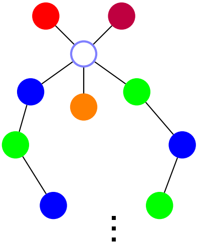

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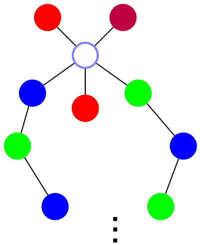

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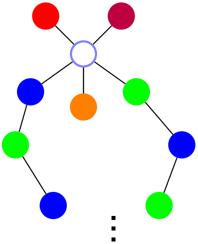

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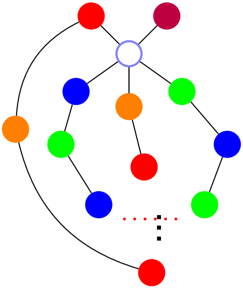

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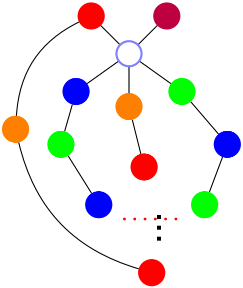

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Planar.

Five color theorem

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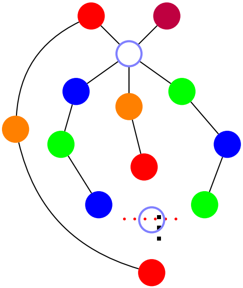
Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



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Switch orange and red in oranges component.
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Planar. $\Longrightarrow$ paths intersect at a vertex!

Five color theorem

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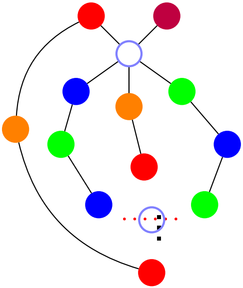

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Planar. $\implies$ paths intersect at a vertex!

What color is it?

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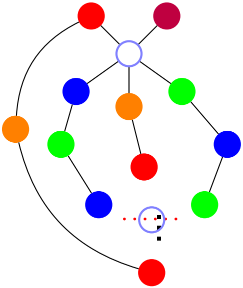

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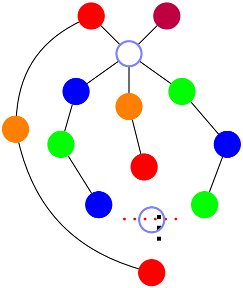

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What color is it?
Must be blue or green to be on that path.

Five color theorem

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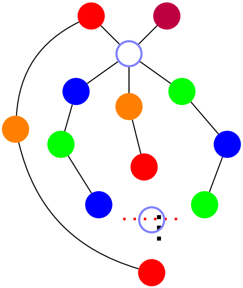

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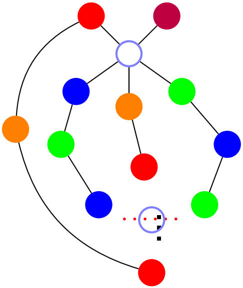

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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

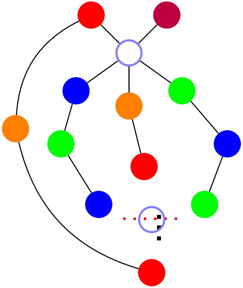
Contradiction.

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



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Done. Unless blue-green path to blue.
Switch orange and red in oranges component.
Done. Unless red-orange path to red.

Planar. $\implies$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.

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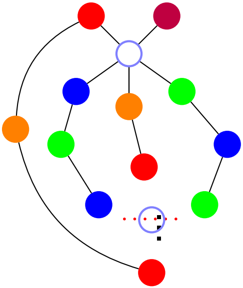

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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

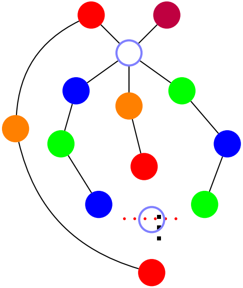
Contradiction. Can recolor one of the neighbors.
Gives an available color for center vertex!

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
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Contradiction. Can recolor one of the neighbors.
Gives an available color for center vertex! □

Four Color Theorem

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Four Color Theorem

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Proof:

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof: Not Today!

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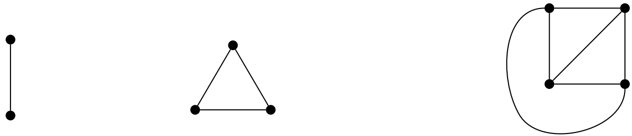
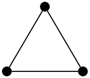
Proof: Not Today!

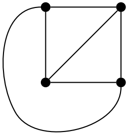
Complete Graph.



K_n complete graph on n vertices.

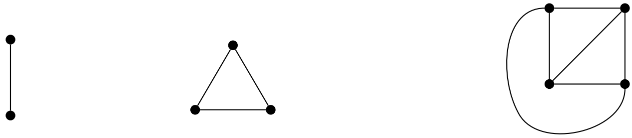
Complete Graph.





K_n complete graph on n vertices.
All edges are present.

Complete Graph.

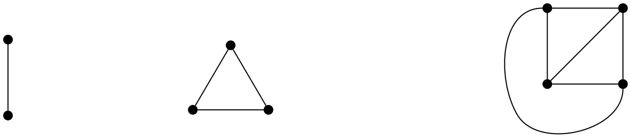


K_n complete graph on n vertices.

All edges are present.

Everyone is my neighbor.

Complete Graph.



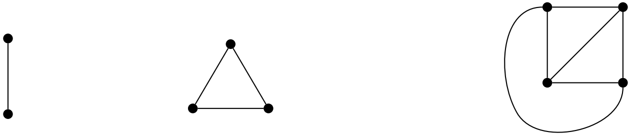
K_n complete graph on n vertices.

All edges are present.

Everyone is my neighbor.

Each vertex is adjacent to every other vertex.

Complete Graph.



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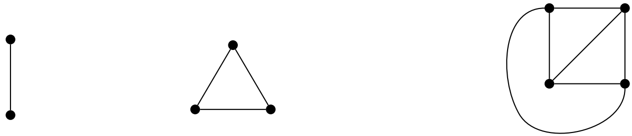

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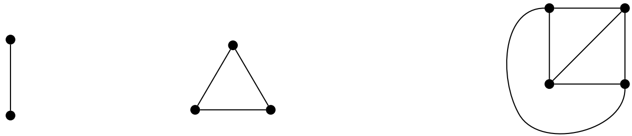
How many edges?

Complete Graph.



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All edges are present.

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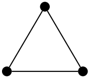
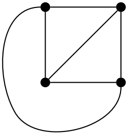
Each vertex is adjacent to every other vertex.

How many edges?

Each vertex is incident to $n-1$ edges.

Complete Graph.





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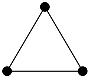
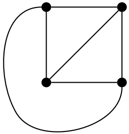
Sum of degrees is $n(n-1)$

Complete Graph.





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All edges are present.

Everyone is my neighbor.

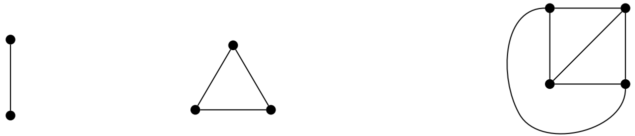
Each vertex is adjacent to every other vertex.

How many edges?

Each vertex is incident to $n-1$ edges.

Sum of degrees is $n(n-1) = 2|E|$

Complete Graph.



K_n complete graph on n vertices.

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Each vertex is adjacent to every other vertex.

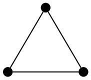
How many edges?

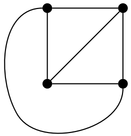
Each vertex is incident to $n-1$ edges.

Sum of degrees is $n(n-1) = 2|E|$

$\implies$ Number of edges is $n(n-1)/2$.

Complete Graph.





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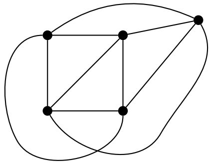
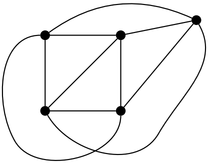
K_4 and K_5



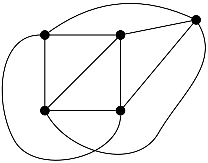
K_5 is not planar.

K_4 and K_5



K_5 is not planar.
Cannot be drawn in the plane without an edge crossing!

K_4 and K_5

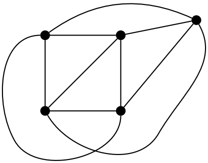


K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it!

K_4 and K_5



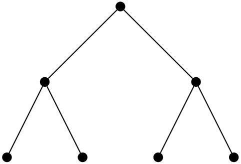
K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it! We did!

A Tree, a tree.

Graph $G = (V, E)$.
Binary Tree!



More generally.

Trees.

Definitions:

Trees.

Definitions:

A connected graph without a cycle.

Trees.

Definitions:

A connected graph without a cycle.
A connected graph with $|V|-1$ edges.

Trees.

Definitions:

A connected graph without a cycle.
A connected graph with $|V|-1$ edges.
A connected graph where any edge removal disconnects it.

Trees.

Definitions:

A connected graph without a cycle.
A connected graph with $|V|-1$ edges.
A connected graph where any edge removal disconnects it.
A connected graph where any edge addition creates a cycle.

Trees.

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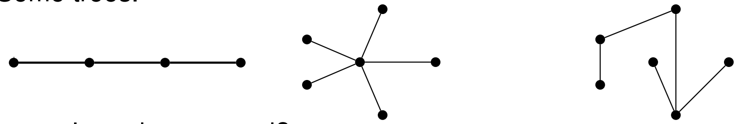
Some trees.



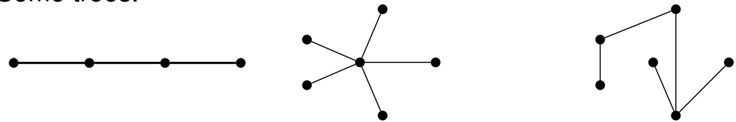
no cycle and connected?

Trees.

Definitions:

A connected graph without a cycle.
A connected graph with $|V| - 1$ edges.
A connected graph where any edge removal disconnects it.
A connected graph where any edge addition creates a cycle.

Some trees.



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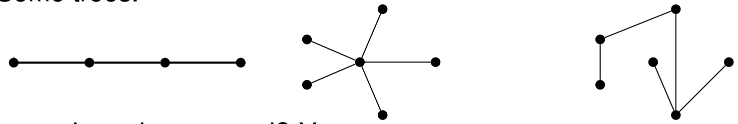

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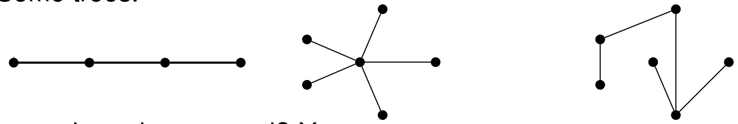

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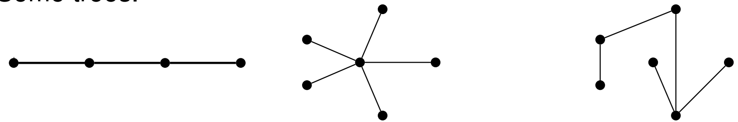

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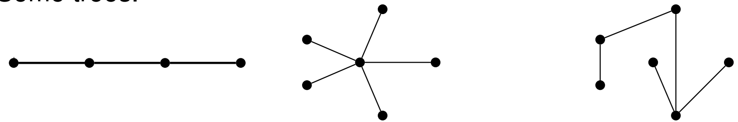

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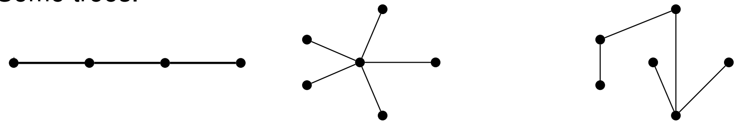

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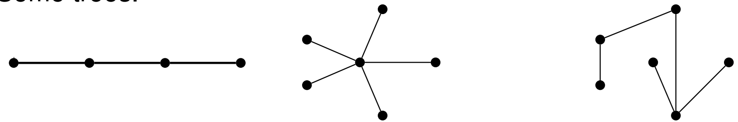

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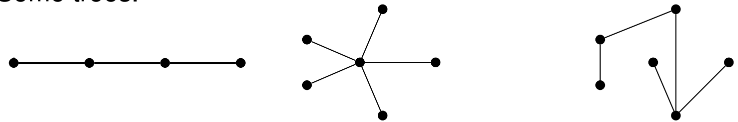

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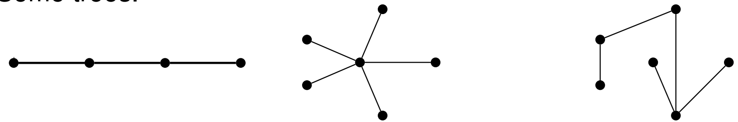

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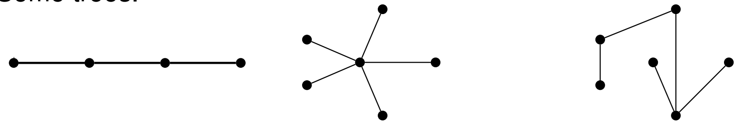

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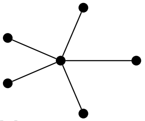
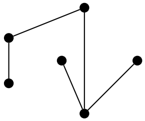

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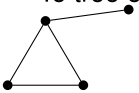
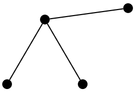
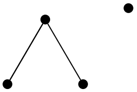
To tree or not to tree!

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Theorem:
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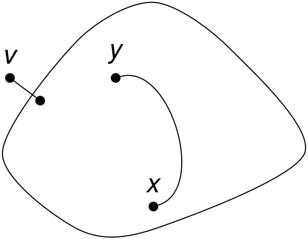

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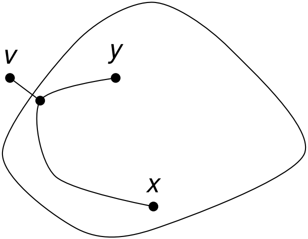

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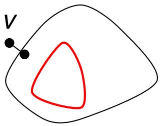

Proof of $\implies$:

Proof of only if.

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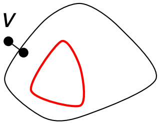
Proof of $\implies$: By induction on $|V|$.



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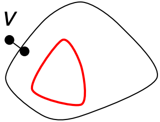

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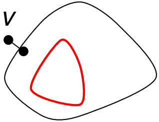

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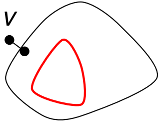

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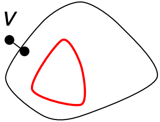

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Claim: There is a degree 1 node.

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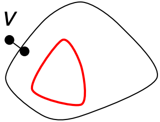

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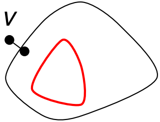

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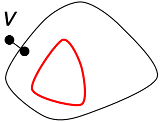
Proof of only if.



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Proof of $\Longrightarrow$: By induction on $|V|$.
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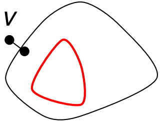
Induction Step:
Claim: There is a degree 1 node.
Proof: First, connected $\Longrightarrow$ every vertex degree ≥ 1.
Sum of degrees is $2|V|-2$
Average degree $2-2/|V|$

Proof of only if.



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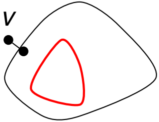
Not everyone is bigger than average!

Proof of only if.



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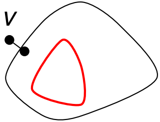

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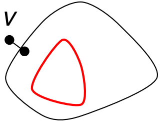

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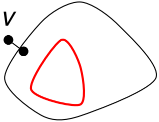

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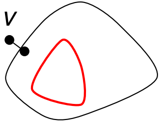

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And no cycle in G since degree 1 cannot participate in cycle.

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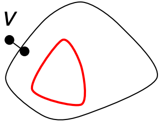

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Thm:
"G is connected and has no cycles"
$\implies$ "G connected and has $|V| - 1$ edges"
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Thm:
"G is connected and has no cycles"
$\implies$ "G connected and has $|V| - 1$ edges"
Proof:
Walk from a vertex using untraversed edges.

Proof of if

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Until get stuck.

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Proof of if

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Until get stuck.
Claim: Degree 1 vertex.
Proof of Claim:
Can't visit more than once since no cycle.

Proof of if

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Removing node doesn't create cycle.

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Removing node doesn't create cycle.
New graph is connected.

Proof of if

Thm:
"G is connected and has no cycles"
$\implies$ "G connected and has $|V| - 1$ edges"
Proof:
Walk from a vertex using untraversed edges.
Until get stuck.
Claim: Degree 1 vertex.
Proof of Claim:
Can't visit more than once since no cycle.
Entered. Didn't leave. Only one incident edge. $\square$
Removing node doesn't create cycle.
New graph is connected.
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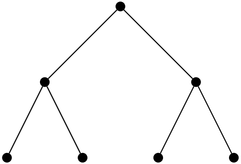
Thm: There is one vertex whose removal disconnects $|V|/2$ nodes from each other.



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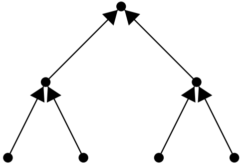

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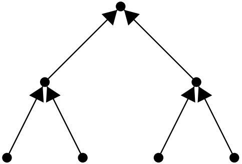

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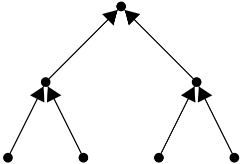

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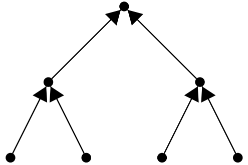

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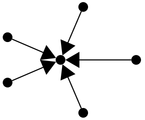

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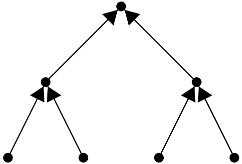

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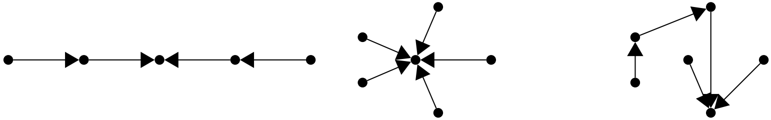

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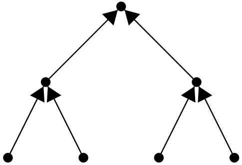

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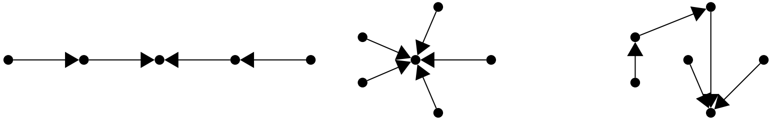

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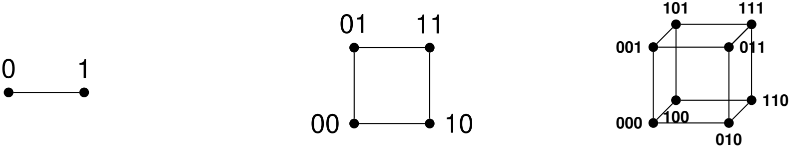

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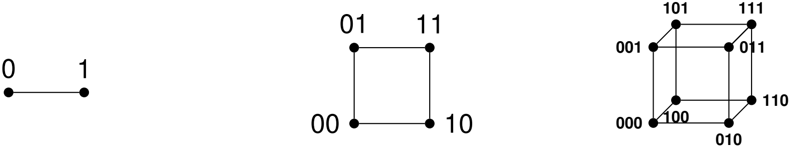

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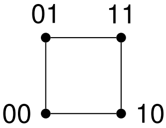

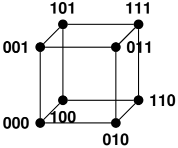

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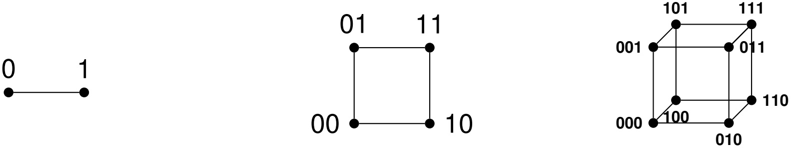

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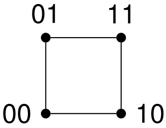

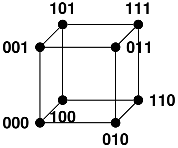

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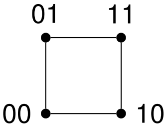

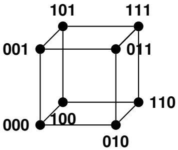

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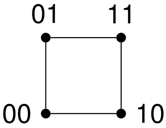

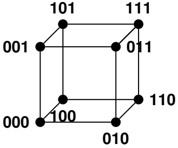

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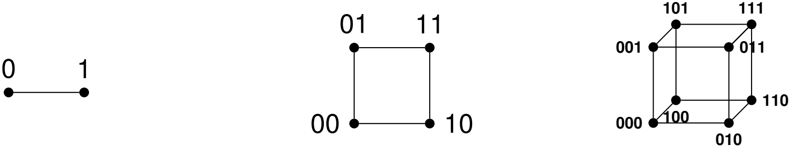

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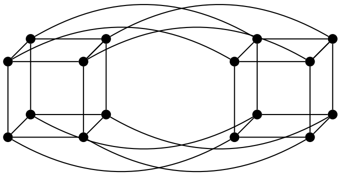

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Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

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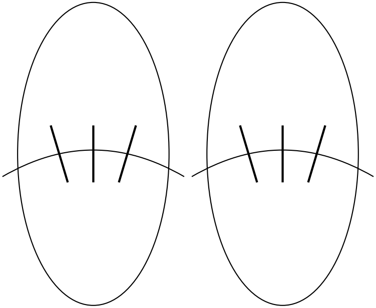

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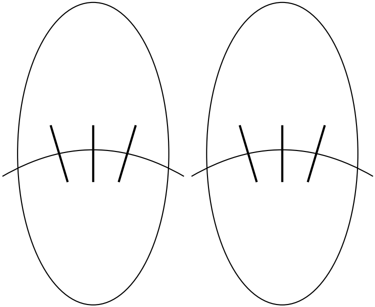

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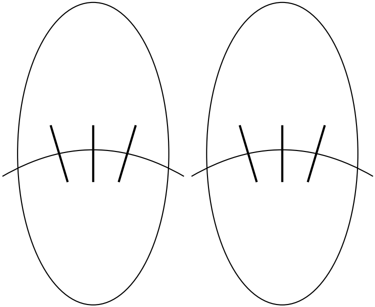

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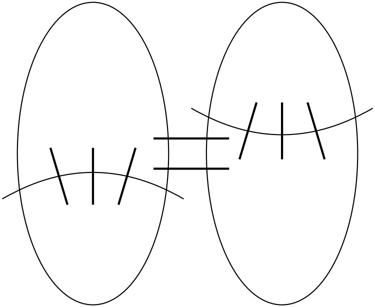

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$H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$

Induction Step

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step.
Recursive definition:
$H_0 = (V_0, E_0)$,$H_1 = (V_1, E_1)$, edges E_x that connect them.
$H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$
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Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$

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Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$
Both S_0 and S_1 are small sides.

Induction Step

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

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Both S_0 and S_1 are small sides. So by induction.

Induction Step

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Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$
Both S_0 and S_1 are small sides. So by induction.
Edges cut in $H_0 \geq |S_0|$.

Induction Step

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Both S_0 and S_1 are small sides. So by induction.
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Edges cut in $H_1 \geq |S_1|$.

Induction Step

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Induction Step

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Both S_0 and S_1 are small sides. So by induction.
Edges cut in $H_0 \geq |S_0|$.
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Total cut edges $\geq |S_0| + |S_1| = |S|$.

Induction Step

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step.
Recursive definition:

$H_0 = (V_0, E_0)$,$H_1 = (V_1, E_1)$, edges E_x that connect them.
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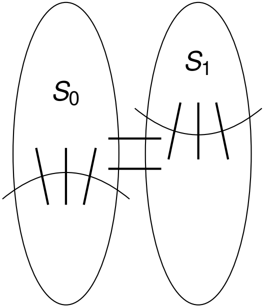
Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$
Both S_0 and S_1 are small sides. So by induction.
Edges cut in $H_0 \geq |S_0|$.
Edges cut in $H_1 \geq |S_1|$.

Total cut edges $\geq |S_0| + |S_1| = |S|$. □

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.
Proof: Induction Step. Case 2.

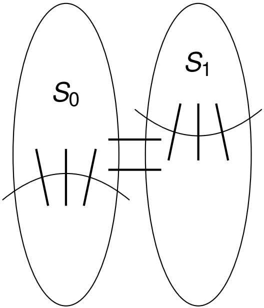
$$|S_0| \geq |V_0|/2.$$



Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



$|S_0| \geq |V_0|/2$.

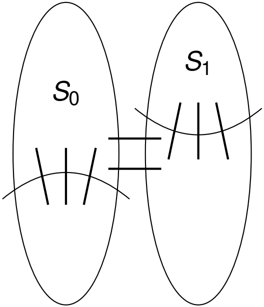
Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.

Induction Step. Case 2.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



$|S_0| \geq |V_0|/2$.

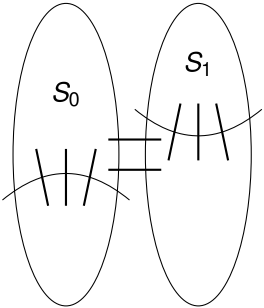
Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.

$\implies \geq |S_1|$ edges cut in E_1.

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.
Proof: Induction Step. Case 2.



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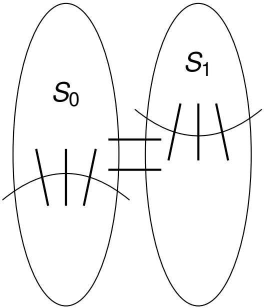
$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.

$\implies \geq |S_1|$ edges cut in E_1.

$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$

Induction Step. Case 2.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.
Proof: Induction Step. Case 2.

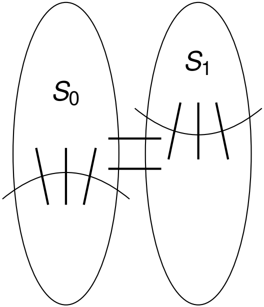


$|S_0| \geq |V_0|/2$.
Recall Case 1: $|S_0|, |S_1| \leq |V|/2$
$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.
$\implies \geq |S_1|$ edges cut in E_1.
$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$
$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



$|S_0| \geq |V_0|/2$.

Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.

$\implies \geq |S_1|$ edges cut in E_1.

$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$

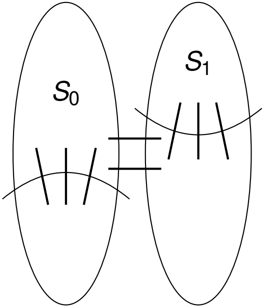
$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Edges in E_x connect corresponding nodes.

Induction Step. Case 2.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



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$\implies \geq |S_1|$ edges cut in E_1.

$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$

$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

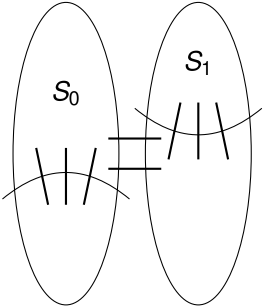
Edges in E_x connect corresponding nodes.

$\implies = |S_0| - |S_1|$ edges cut in E_x.

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



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$\implies \geq |S_1|$ edges cut in E_1.

$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$

$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

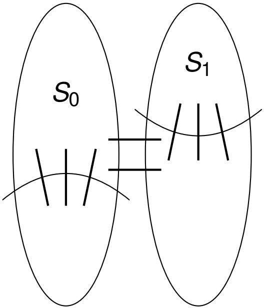
Edges in E_x connect corresponding nodes.

$\implies = |S_0| - |S_1|$ edges cut in E_x.

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



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$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$

$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Edges in E_x connect corresponding nodes.

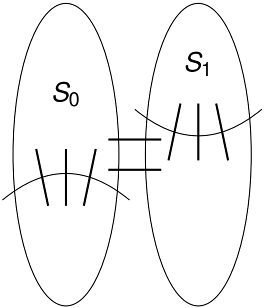
$\implies = |S_0| - |S_1|$ edges cut in E_x.

Total edges cut:

Induction Step. Case 2.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



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$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Edges in E_x connect corresponding nodes.

$\implies = |S_0| - |S_1|$ edges cut in E_x.

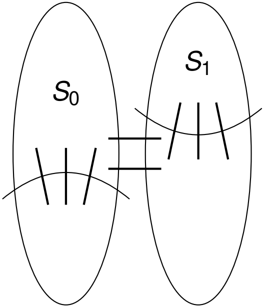
Total edges cut:

$\geq$

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



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Edges in E_x connect corresponding nodes.

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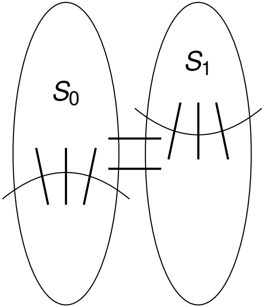
Total edges cut:

$\geq \ |S_1|$

Induction Step. Case 2.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



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$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Edges in E_x connect corresponding nodes.

$\implies = |S_0| - |S_1|$ edges cut in E_x.

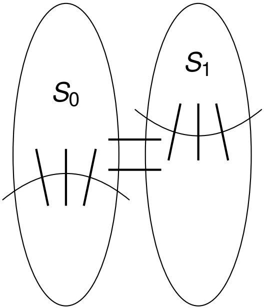
Total edges cut:

$\geq \ |S_1| + |V_0| - |S_0|$

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



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Edges in E_x connect corresponding nodes.

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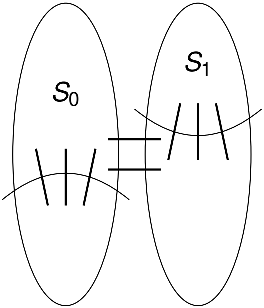
Total edges cut:

$\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1|$

Induction Step. Case 2.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



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Edges in E_x connect corresponding nodes.

$\implies = |S_0| - |S_1|$ edges cut in E_x.

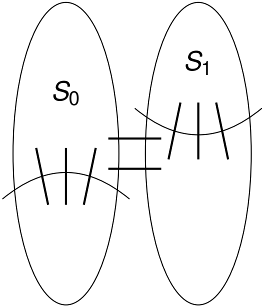
Total edges cut:

$\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$

Induction Step. Case 2.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



$|S_0| \geq |V_0|/2$.

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Edges in E_x connect corresponding nodes.

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Total edges cut:

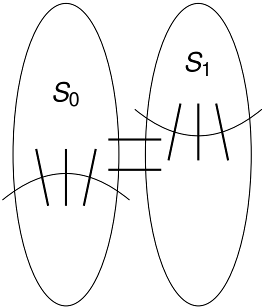
$\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$

$|V_0|$

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



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$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Edges in E_x connect corresponding nodes.

$\implies = |S_0| - |S_1|$ edges cut in E_x.

Total edges cut:

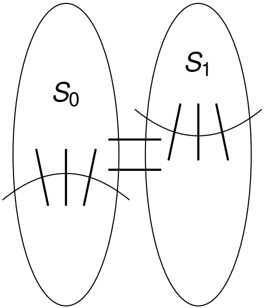
$\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$

$|V_0| = |V|/2 \geq |S|$.

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



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$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$

$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Edges in E_x connect corresponding nodes.

$\implies = |S_0| - |S_1|$ edges cut in E_x.

Total edges cut:

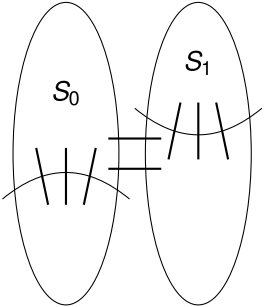
$\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$

$|V_0| = |V|/2 \geq |S|$. □

Induction Step. Case 2.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2.



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$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Edges in E_x connect corresponding nodes.

$\implies = |S_0| - |S_1|$ edges cut in E_x.

Total edges cut:

$\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$

$|V_0| = |V|/2 \geq |S|$. □

Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

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Have a nice weekend!