Today.

Comment: Add 0.

Add $(k-k)$.

Induction: Some quibbles.

Induction and Recursion

Couple of more induction proofs.

Stable Marriage.

Some quibbles.

The induction principle works on the natural numbers.

Proves statements of form: $\forall n \in \mathbb{N}, P(n)$.

Yes.

What if the statement is only for $n \geq 3$?

$\forall n \in \mathbb{N}, (n \geq 3) \implies P(n)$

Restate as:
$\forall n \in \mathbb{N}, Q(n)$ where $Q(n) =$ "$(n \geq 3) \implies P(n)$".

Base Case: typically start at 3.
Since $\forall n \in \mathbb{N}, Q(n) \implies Q(n+1)$ is trivially true before 3.

Can you do induction over other things? Yes.

Any set where any subset of the set has a smallest element.

In some sense, the natural numbers.

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Base cases: P(12) , P(13) , P(14) , P(15). Yes.

Strong Induction step:
Recursive call is correct: $P(n-4) \implies P(n)$.
$n-4 = 4x' + 5y' \implies n = 4(x'+1) + 5(y')$

Slight differences: showed for all $n \geq 16$ that $\wedge_{i=4}^{n-1} P(i) \implies P(n)$.

Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. $1 \leq 2$.
Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$.

$$\sum_{i=1}^{k+1} \frac{1}{i^2}$$
$$= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.$$
$$\leq 2 + \frac{1}{(k+1)^2}$$

Uh oh?

Hmmm... It better be that any sum is *strictly less than* 2.

How much less? At least by $\frac{1}{(k+1)^2}$ for S_k.

"$S_k \leq 2 - \frac{1}{(k+1)^2}$" $\Longrightarrow$ "$S_{k+1} \leq 2$"
Induction step works! No! Not the same statement!!!!
Need to prove "$S_{k+1} \leq 2 - \frac{1}{(k+2)^2}$".

Darn!!!

Strenthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:

Ind hyp: $P(k)$ — "$S_k \leq 2 - f(k)$"

Prove: $P(k+1)$ – "$S_{k+1} \leq 2 - f(k+1)$"

$$\begin{aligned} S(k+1) &= S_k + \frac{1}{(k+1)^2} \\ &\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.} \end{aligned}$$

Choose $f(k+1) \leq f(k) - \frac{1}{(k+1)^2}$.

$$\implies S(k+1) \leq 2 - f(k+1).$$

Can you?

Subtracting off a quadratically decreasing function every time.

Maybe a linearly decreasing function to keep positive?

Try $f(k) = \frac{1}{k}$

$$\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2}?$$

$$1 \leq \frac{k+1}{k} - \frac{1}{k+1} \quad \text{Multiplied by } k+1.$$

$$1 \leq 1 + \left(\frac{1}{k} - \frac{1}{k+1}\right) \quad \text{Some math. So yes!}$$

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.

Stable Marriage Problem

- Small town with n boys and n girls.
- Each girl has a ranked preference list of boys.
- Each boy has a ranked preference list of girls.

How should they be matched?

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

Uh..oh.

So..

Produce a pairing where there is no running off!

Definition: A **pairing** is disjoint set of n boy-girl pairs.

Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

Definition: A **rogue couple** b, g^* for a pairing S:
b and g^* prefer each other to their partners in S

Example: Brad and Angelina are a rogue couple in S.

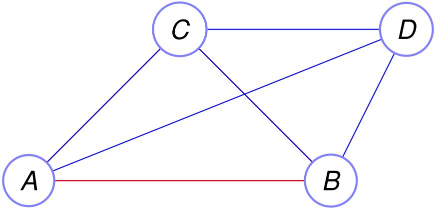
A stable pairing??

Given a set of preferences.

Is there a stable pairing?
How does one find it?

Consider a single gender version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



The Traditional Marriage Algorithm.

Each Day:

1. Each boy **proposes** to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer
   (whom she puts on a string.)
3. Rejected boy crosses rejecting girl off his list.

Stop when each girl gets exactly one proposal.
Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do “better”?

Example.

Boys			
A	~~1~~	2	3
B	~~1~~	~~2~~	3
C	~~2~~	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~, C	C	C
2	C	B, ~~C~~	B	A, ~~B~~	A
3					B

Termination.

Every non-terminated day a boy **crossed** an item off the list.

Total size of lists? n boys, n length list. n^2

Terminates in at most n^2+1 steps!

It gets better every day for girls..

Improvement Lemma: It just gets better for girls.
If on day t a girl g has a boy b on a string,
any boy, b', on g's string for any day $t' > t$
is at least as good as b.

Let's apply lemma.

Girl "Alice" has boy "Bob" on string on day 5.

She has boy "Jim" on string on day 7.

Does Alice prefer "Jim" or "Bob"?

g - 'Alice', b - 'Bob', b' - 'Jim', $t = 5$, $t' = 7$.

Improvement Lemma says she prefers 'Jim'.

On day 10, could Alice still have "Jim" on her string? Yes.

She likes her day 10 boy at least as much as her day 7 boy.
Here, $b = b'$.

Why is lemma true?

Proof Idea: Because she can always keep the previous boy on the string.

Improvement Lemma

Improvement Lemma: It just gets better for girls.

If on day t a girl g has a boy b on a string, any boy, b', on g's string for any day $t' > t$ is at least as good as b.

Proof:

$P(k)$- - "boy on g's string is at least as good as b on day $t+k$"

$P(0)$– true. Girl has b on string.

Assume $P(k)$. Let b' be boy **on string** on day $t+k$.

On day $t+k+1$, boy b' comes back.
Girl can choose b', or do better with another boy, b''

That is, $b' \leq b$ by induction hypothesis.
And b'' is better than b' by algorithm.
$\implies$ Girl does at least as well as with b.

$P(k) \implies P(k+1)$. And by principle of induction. $\square$

Pairing when done.

Lemma: Every boy is matched at end.

Proof:
If not, a boy b must have been rejected n times.

Every girl has been proposed to by b,
and Improvement lemma

$\implies$ each girl has a boy on a string.

and each boy is on at most one string.

n girls and n boys. Same number of each.

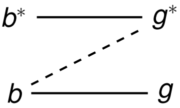
$\implies$ b must be on some girl's string!

Contradiction. $\square$

Pairing is Stable.

Lemma: There is no rogue couple for the pairing formed by traditional marriage algorithm.

Proof:
Assume there is a rogue couple; (b, g^*)



b likes g^* more than g.

g^* likes b more than b^*.

Boy b proposes to g^* before proposing to g.

So g^* rejected b (since he moved on)

By improvement lemma, g^* likes b^* better than b.

Contradiction! $\square$

Good for boys? girls?

Is the TMA better for boys? for girls?

Definition: A **pairing is x-optimal** if $x's$ partner
is its best partner in any stable pairing.

Definition: A **pairing is x-pessimal** if $x's$ partner
is its worst partner in any stable pairing.

Definition: A **pairing is boy optimal** if it is x-optimal for **all** boys x.

..and so on for boy pessimal, girl optimal, girl pessimal.

Claim: The optimal partner for a boy must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any stable pairing.
As well as you can be in a globally stable solution!

Question: Is there a boy or girl optimal pairing?
Is it possible:
b-optimal pairing different from the b'-optimal pairing!
Yes? No?

Understanding Optimality: by example.

A:	1,2	1:	A,B
B:	1,2	2:	B,A

Consider pairing: $(A,1),(B,2)$.

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A:	1,2	1:	B,A
B:	2,1	2:	A,B

Pairing S: $(A,1),(B,2)$. Stable? Yes.

Pairing T: $(A,2),(B,1)$. Also Stable.

Which is optimal for A? S Which is optimal for B? S
Which is optimal for 1? T Which is optimal for 2? T

TMA is optimal!

For boys? For girls?

Theorem: TMA produces a boy-optimal pairing.

Proof:
Assume not: there is a boy b who do not get their optimal girl, g.

There is a stable pairing S where b and g are paired.

Let t be first day a boy b gets rejected
by his optimal girl g who he is paired with
in stable pairing S.

b^* - knocks b off of g's string on day $t \implies g$ prefers b^* to b

By choice of t, b^* prefers g to optimal girl.

$\implies b^*$ prefers g to his partner g^* in S.

Rogue couple for S.
So S is not a stable pairing. Contradiction. $\square$

Notes: S - stable. $(b^*, g^*) \in S$. But (b^*, g) is rogue couple!

Used Well-Ordering principle...Induction.

How about for girls?

Theorem: TMA produces girl-pessimal pairing.

T – pairing produced by TMA.

S – worse stable pairing for girl g.

In T, (g, b) is pair.

In S, (g, b^*) is pair.

g likes b^* less than she likes b.

T is boy optimal, so b likes g more than his partner in S.

(g, b) is Rogue couple for S

S is not stable.

Contradiction. $\square$

Notes: Not really induction.
Structural statement: Boy optimality $\Longrightarrow$ Girl pessimality.

Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.
TMA - boys propose.
Girls could propose. $\implies$ optimal for girls.

Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

..until 1990's...Resident optimal.

Another variation: couples.

Don’t go!

Summary.

▸ Link