Today.

Principle of Induction.(continued.)

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$$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

And we get...

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Climb an infinite ladder?

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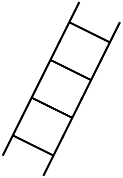
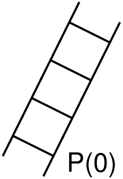


Climb an infinite ladder?

$P(0)$



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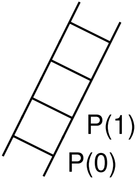
$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$



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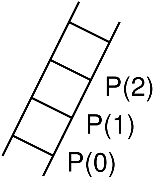

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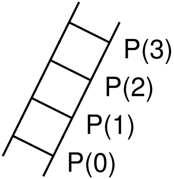

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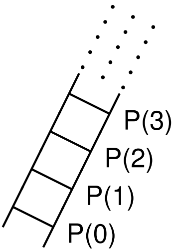

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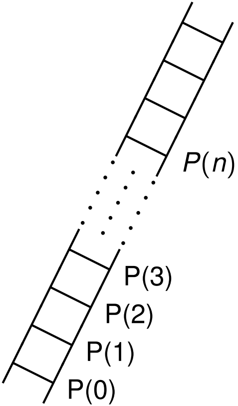

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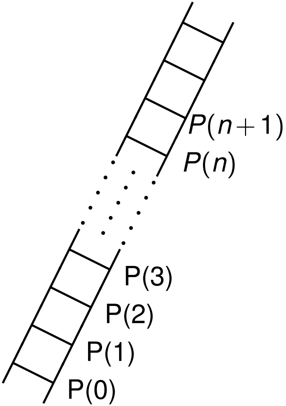

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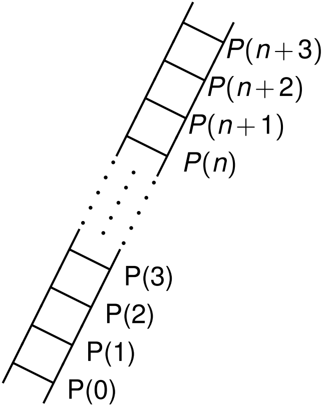

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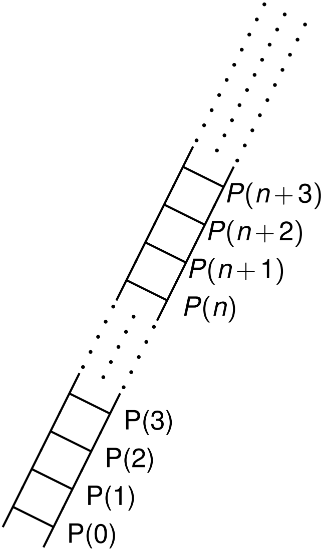

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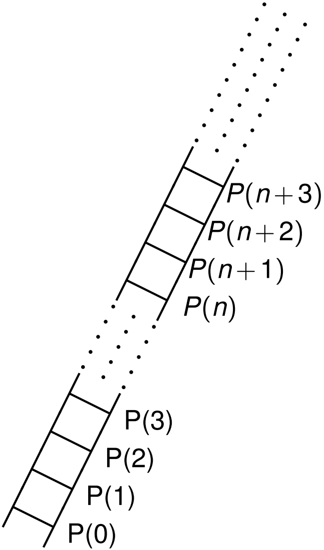

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Your favorite example of forever..

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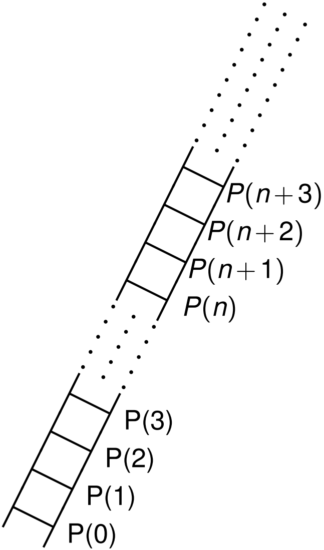

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Your favorite example of forever..or the natural numbers...

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$

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How about $k+2$.

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Is this a proof? It shows that we can always move to the next step.

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Gauss and Induction

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Gauss and Induction

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plus inductive step

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

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How about $k+2$. Same argument starting at $k+1$ works!

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Statement is true for $n = 0$ $P(0)$ is true

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Predicate, $P(n)$, True for all natural numbers! **Proof by Induction.**

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

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$(k+1)^3 - (k+1)$

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$(q + k^2 + k)$ is integer (closed under addition and multiplication).

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&= 3(q + k^2 + k) \qquad \text{(Un)Distributive + over} \times
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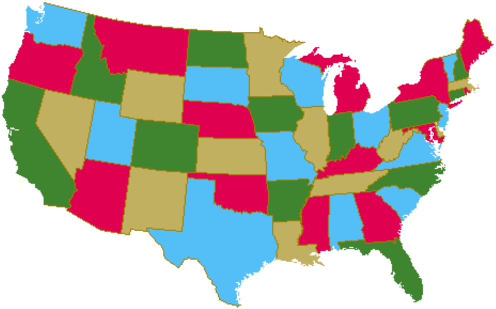
Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.



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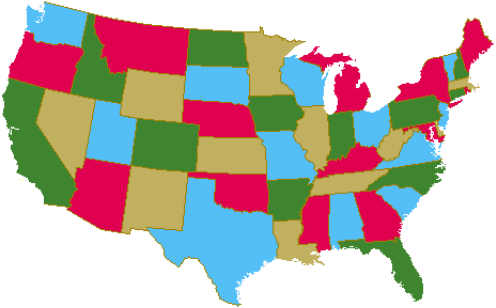

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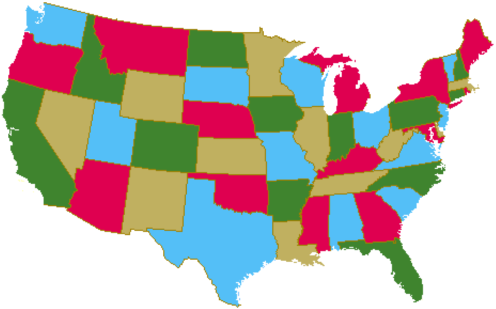

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States connected at a point, can have same color.

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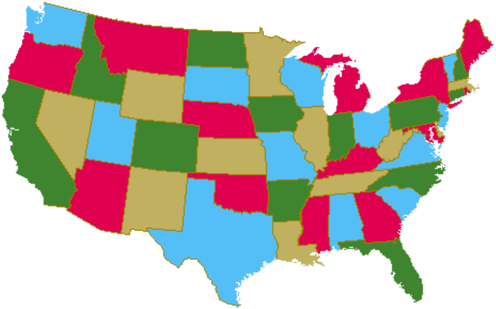

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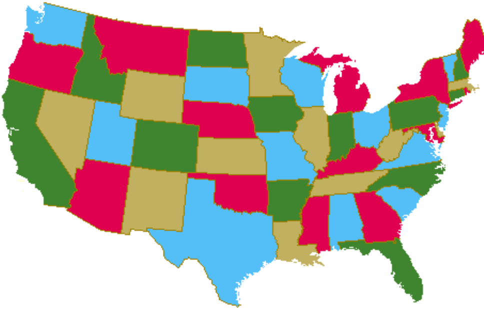

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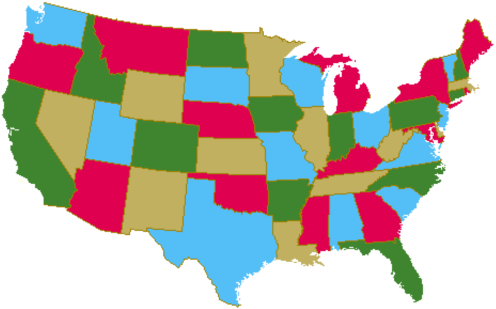

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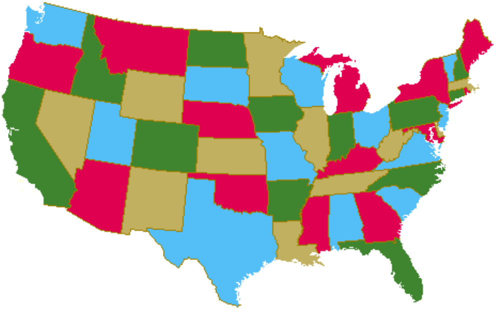

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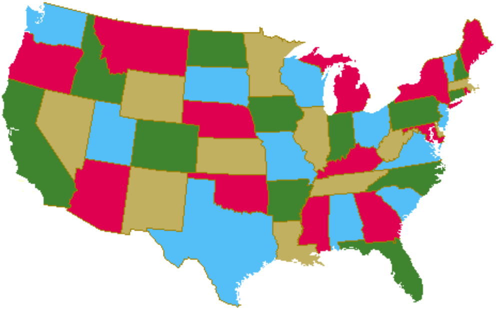

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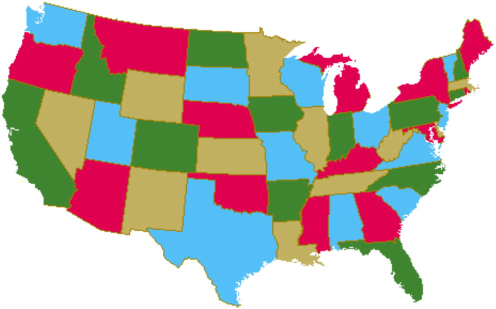

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Two color theorem: example.

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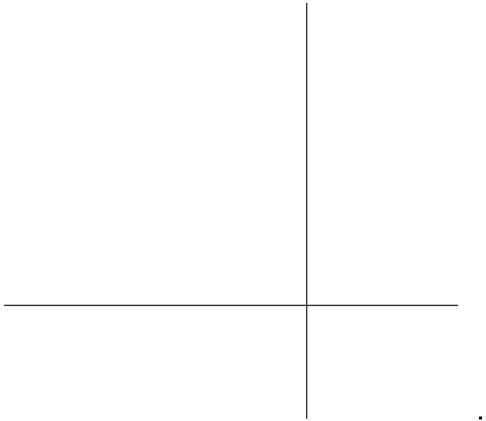

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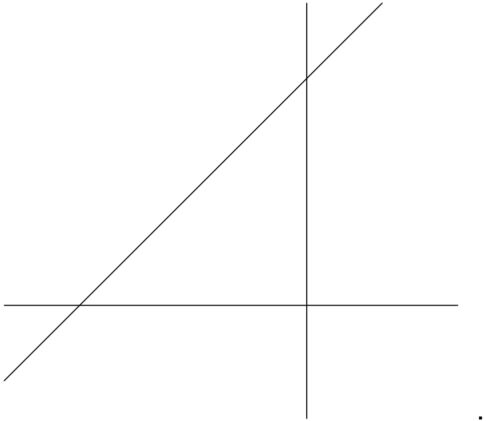

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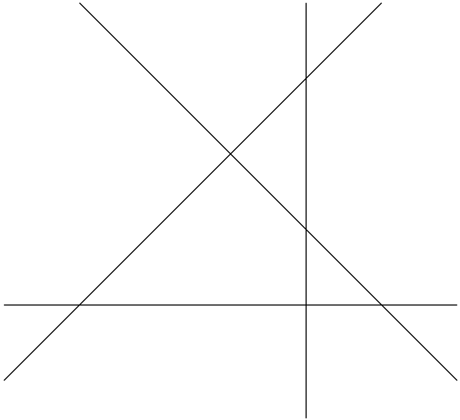

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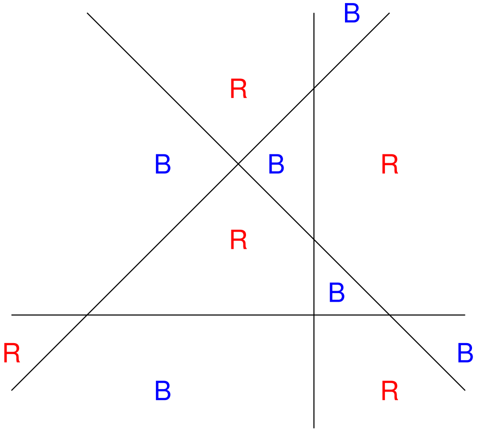

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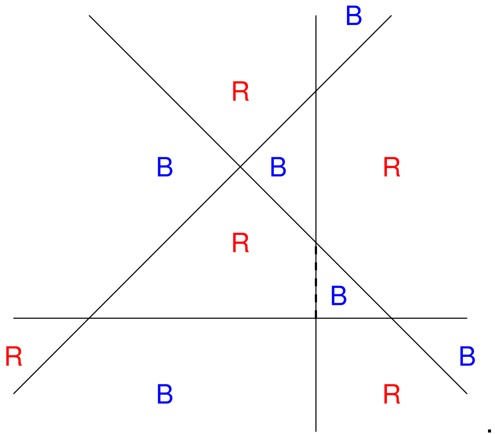

Fact: Swapping red and blue gives another valid colors.

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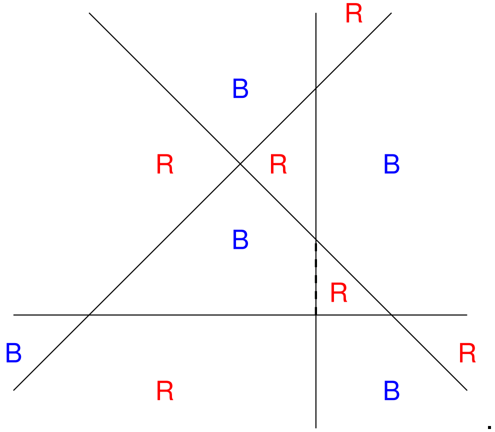

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Two color theorem: proof illustration.

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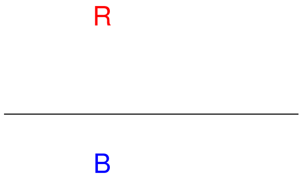

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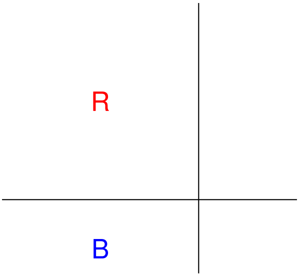

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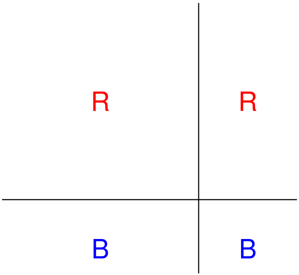
Two color theorem: proof illustration.



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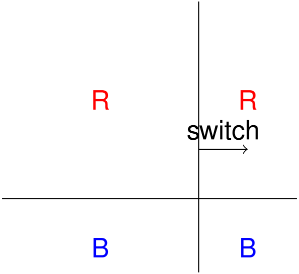

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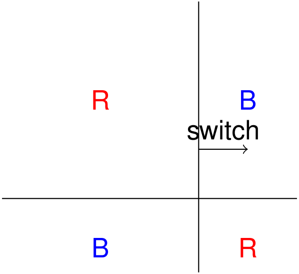

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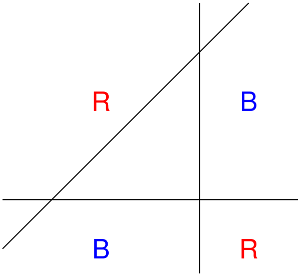

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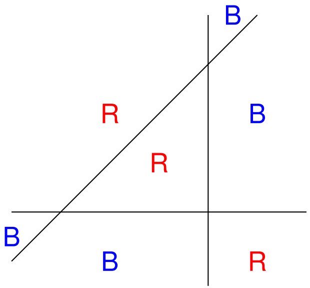

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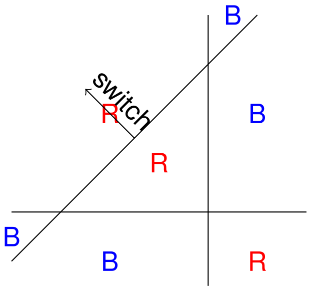

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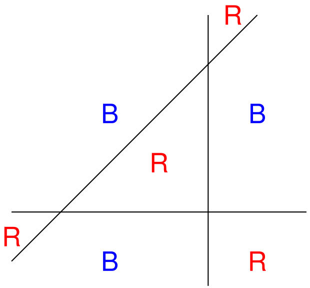

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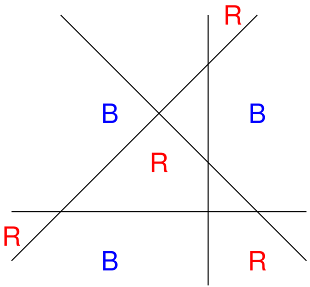

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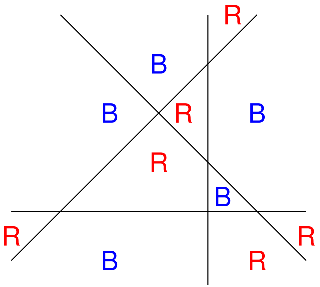

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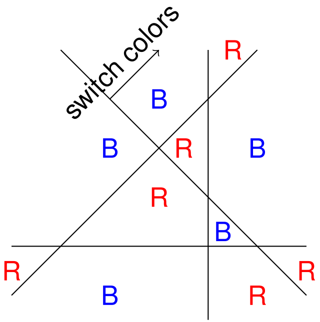

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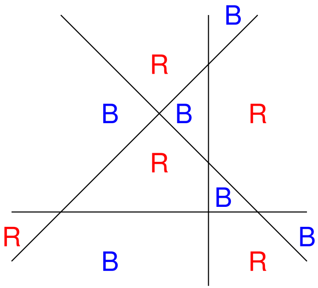

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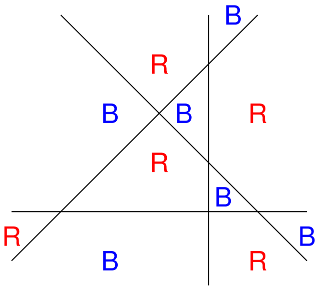

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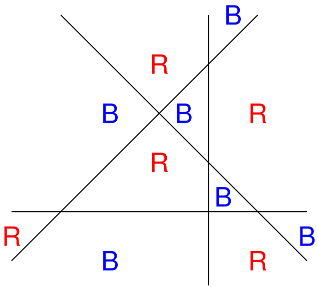

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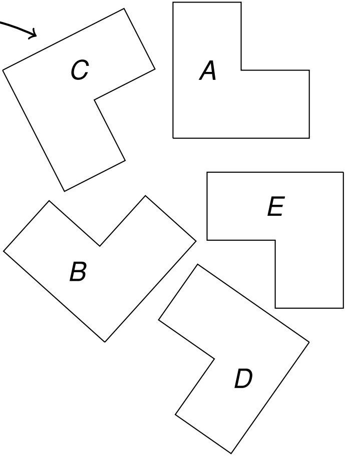
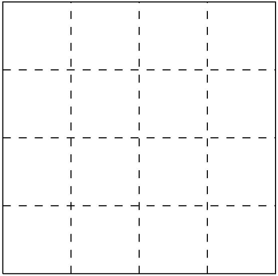
□

Tiling Cory Hall Courtyard.

Use these L-tiles.



To Tile this 4×4 courtyard.



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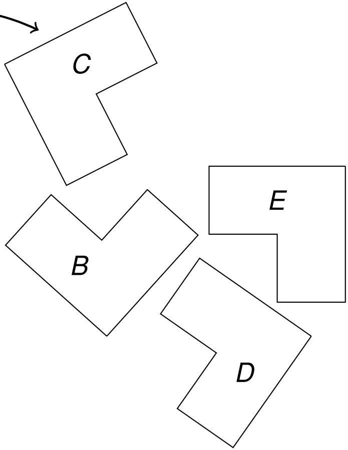

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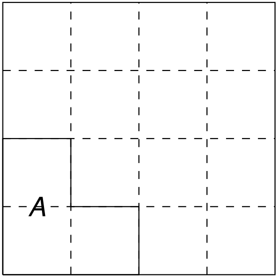
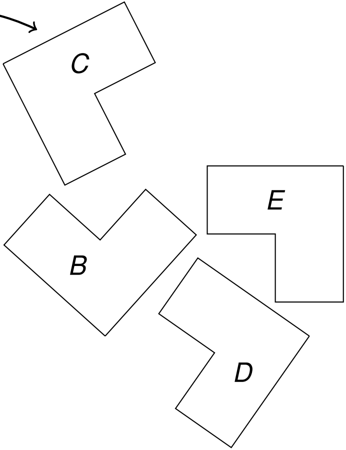


Tiling Cory Hall Courtyard.

Use these L-tiles.



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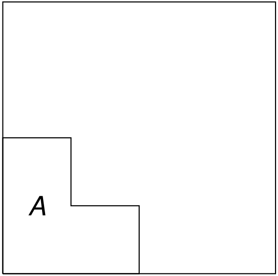

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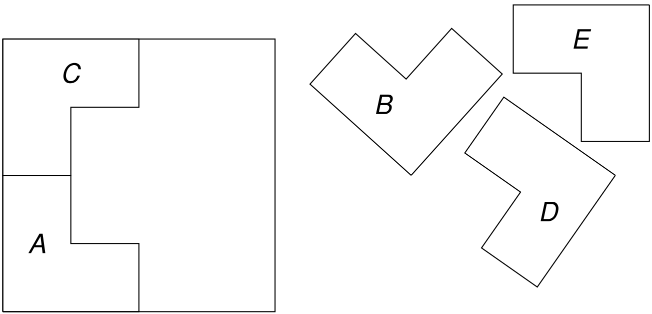

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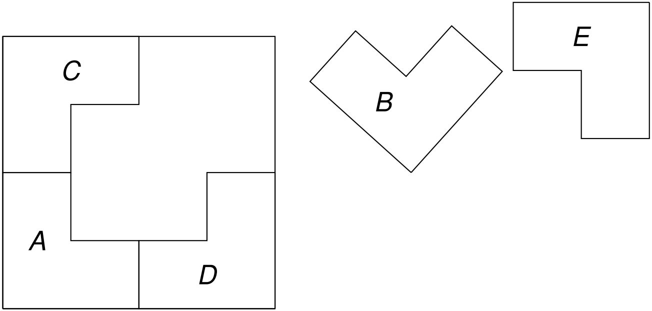

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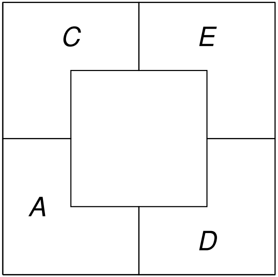

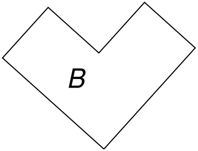

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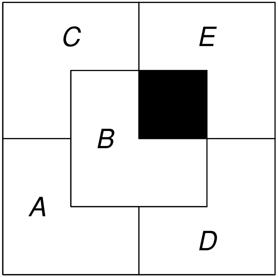

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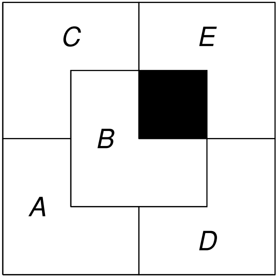

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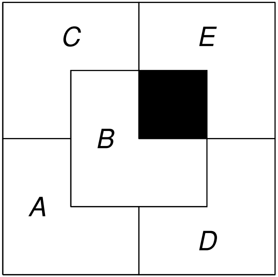


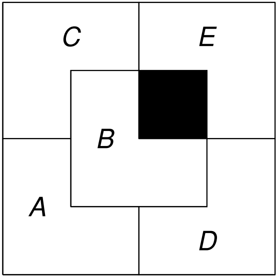
Alright!

Tiled 4×4 square with 2×2 L-tiles.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Tiling Cory Hall Courtyard.

Use these L-tiles.

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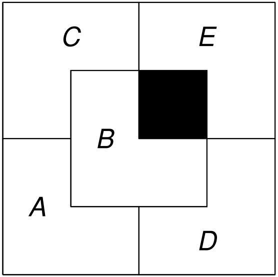

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Can we tile any $2^n \times 2^n$ with L-tiles (with a hole)

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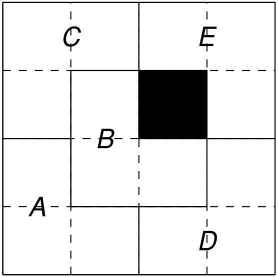

Alright!

Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n!

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

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Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

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$$2^{2(k+1)}$$

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Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.
Proof:

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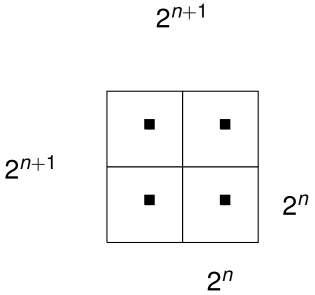

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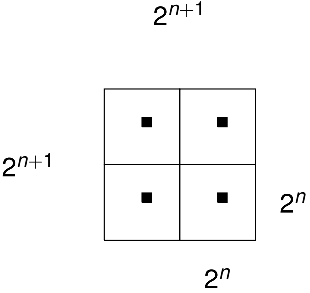

What to do now???

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Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

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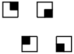

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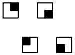
Flipping the orientation can leave hole anywhere. 

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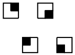

Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.

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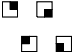

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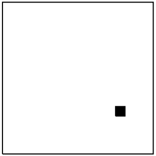

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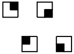
Flipping the orientation can leave hole anywhere. 

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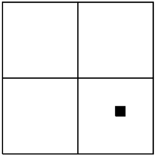

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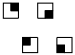
Flipping the orientation can leave hole anywhere. 

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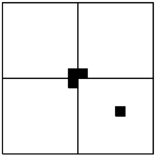

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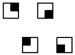

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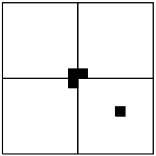

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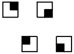

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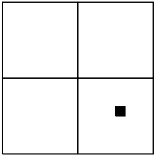

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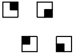

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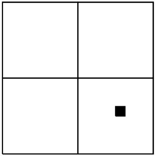

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Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

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Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

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E.g. Reduced form is "smallest" form of rational number a/b.

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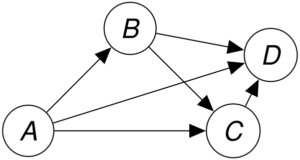

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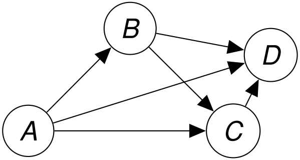

Theorem: Any tournament that has a cycle has a cycle of length 3.

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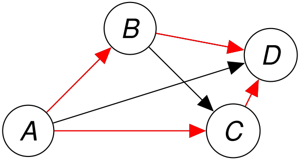

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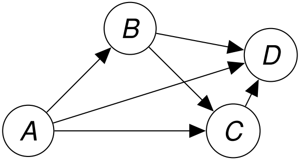

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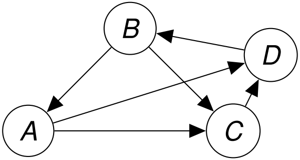

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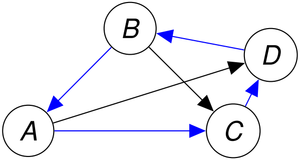

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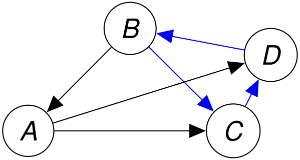

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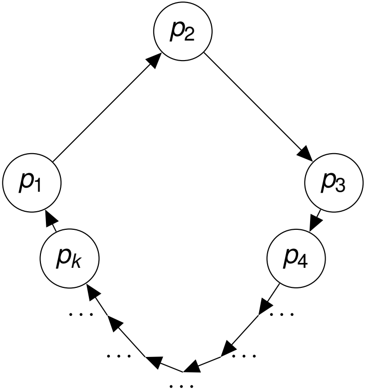

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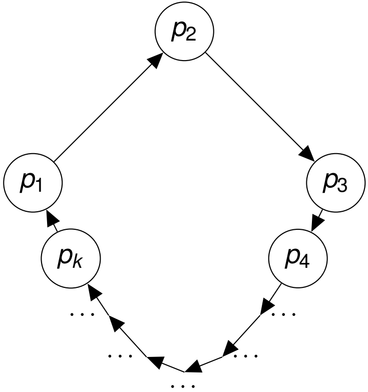

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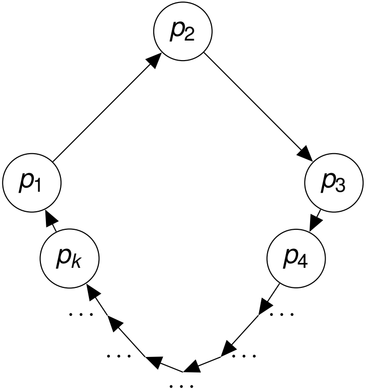

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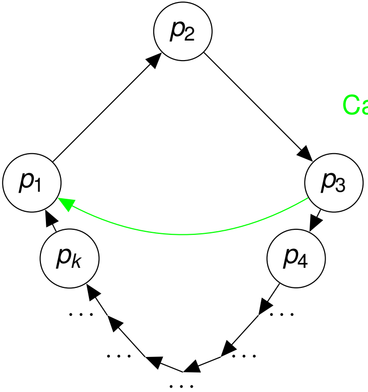

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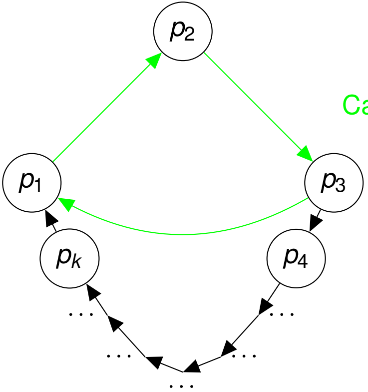

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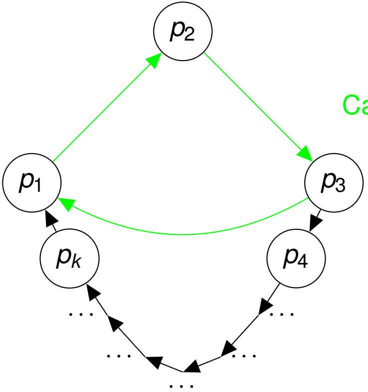

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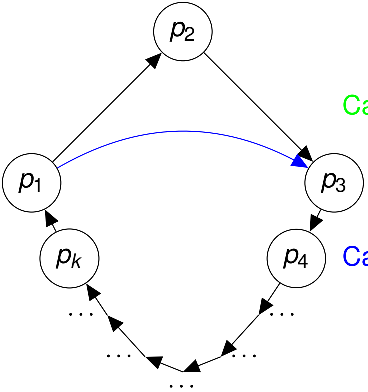

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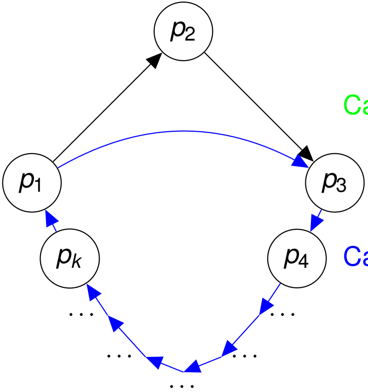

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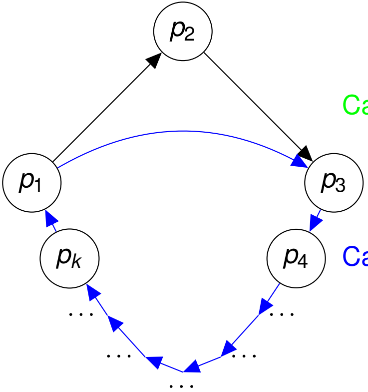

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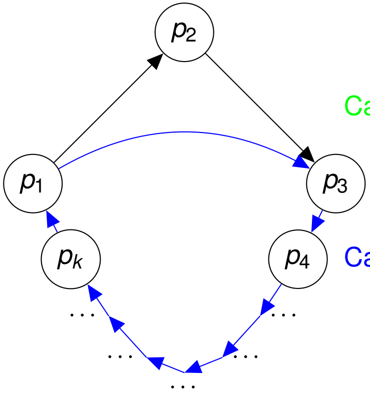

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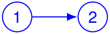

Tournaments have long paths.

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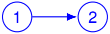

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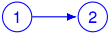

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Tournament on $n+1$ people,

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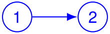

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Tournament on $n+1$ people,
Remove arbitrary person

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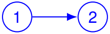

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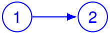

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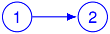

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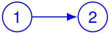

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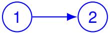

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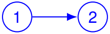

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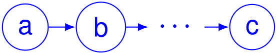

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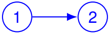

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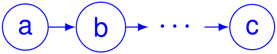

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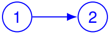

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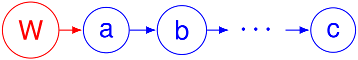

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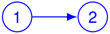

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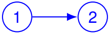

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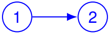

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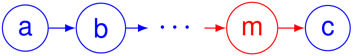

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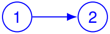

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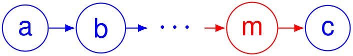

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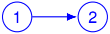

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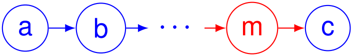

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□

Horses of the same color...

Theorem: All horses have the same color.

Horses of the same color...

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Base Case: $P(1)$ - trivially true.

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A horse in the middle in common! $1, 2, 3, \ldots, k, k+1$

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All k must have the same color.	$1, 2, 3, \ldots, k, k+1$

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Fix base case.

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Of course it doesn't work.

As we will see, it is more subtle to catch errors in proofs of correct theorems!!

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

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Any islander who knows they have green eyes must commit ritual suicide that day.

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No islander knows there own eye color, but knows everyone elses.

Sad Islanders...

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Visitor: "I see someone has green eyes."

Result: On day 100, they all do the ritual.

Why?

They know induction.

Thm: If there are n villagers with green eyes they do ritual on day n.

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Proof:
Base: $n = 1$. Person with green eyes does ritual on day 1.

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Induction hypothesis:

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Thm: If there are n villagers with green eyes they do ritual on day n.

Proof:
Base: $n = 1$. Person with green eyes does ritual on day 1.

Induction hypothesis:
If there were n people with green eyes, they would do ritual on day n.

They know induction.

Thm: If there are n villagers with green eyes they do ritual on day n.

Proof:
Base: $n = 1$. Person with green eyes does ritual on day 1.

Induction hypothesis:
If there were n people with green eyes, they would do ritual on day n.

Induction step:
On day $n+1$, a green eyed person sees n people with green eyes.

They know induction.

Thm: If there are n villagers with green eyes they do ritual on day n.

Proof:
Base: $n = 1$. Person with green eyes does ritual on day 1.

Induction hypothesis:
If there were n people with green eyes, they would do ritual on day n.

Induction step:
On day $n + 1$, a green eyed person sees n people with green eyes.

But they didn't do the ritual.

They know induction.

Thm: If there are n villagers with green eyes they do ritual on day n.

Proof:
Base: $n = 1$. Person with green eyes does ritual on day 1.

Induction hypothesis:
If there were n people with green eyes, they would do ritual on day n.

Induction step:
On day $n+1$, a green eyed person sees n people with green eyes.

But they didn't do the ritual.

So there must be $n+1$ people with green eyes.

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One of them, is me.

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Wait! Visitor added no information.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

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On day 1, everyone knows everyone sees more than zero.

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On day 99, no one sees 98

Common Knowledge.

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On day 99, no one sees 98 since everyone knows everyone else does not see 97...

Common Knowledge.

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On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100,

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

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On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

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On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

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On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:
Emperor's new clothes!

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

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On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:
Emperor's new clothes!
No one knows other people see that he has no clothes.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

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On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:
Emperor's new clothes!
  No one knows other people see that he has no clothes.
  Until kid points it out.

Summary: principle of induction.

Today: More induction.

Summary: principle of induction.

Today: More induction.

$(P(0)$

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1))))$

Summary: principle of induction.

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$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

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$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0

Summary: principle of induction.

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Base Case: Prove $P(n_0)$.

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Base Case: Prove $P(n_0)$.
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Statement is proven!

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Strong Induction:

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Strong Induction:
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Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n)) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Summary: principle of induction.

Today: More induction.

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Also Today: strengthened induction hypothesis.

Strengthen theorem statement.

Summary: principle of induction.

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Also Today: strengthened induction hypothesis.

Strengthen theorem statement.
Sum of first n odds is n^2.

Summary: principle of induction.

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Hole anywhere.

Summary: principle of induction.

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Not same as strong induction. E.g., used in product of primes proof.

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Induction $\equiv$ Recursion.

Summary: principle of induction.

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Variations:

Strong Induction:

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Summary: principle of induction.

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Variations:
Strong Induction:
$$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$$

Different Starting Point:
$$(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1))))$$

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