Today.

Principle of Induction (continued.)

\[ P(0) \land (\forall n \in \mathbb{N}) P(n) \implies P(n+1) \]

And we get...

\[ (\forall n \in \mathbb{N}) P(n) \]

...Yes for 0, and we can conclude Yes for 1...

and we can conclude Yes for 2......

Another Induction Proof.

Theorem: For every \( n \in \mathbb{N} \), \( n^2 - n \) is divisible by 3. \( (3|n^2 - n) \).

Proof: By induction.

Base Case: \( P(0) \) is \( (0^2 - 0) \) divisible by 3. Yes!

Induction Step: \( (\forall k \in \mathbb{N}) P(k) \implies P(k + 1) \)

Induction Hypothesis: \( k^2 - k \) is divisible by 3.

or \( k^2 - k = 3q \) for some integer \( q \).

\[
(k + 1)^2 - (k + 1) = k^2 + 3k^2 + 3k + 1 - (k + 1)
\]

\[
= k^2 + 3k^2 + 2k
\]

\[
= (k^2 - k) + 3k^2 + 3k
\]

Subtract/add \( k \)

\[
= 3q + 3(k^2 + k)
\]

Induction Hyp. Factor.

\[
= 3(q + k^2 + k)
\]

(Un)distributive + over \( \times \)

Or \( (k + 1)^2 - (k + 1) = 3(q + k^2 + k) \).

\( q + k^2 + k \) is integer (closed under addition and multiplication).

\( \implies (k + 1)^2 - (k + 1) \) is divisible by 3.

Thus, \( (\forall k \in \mathbb{N}) P(k) \implies P(k + 1) \)

Thus, theorem holds by induction.


Climb an infinite ladder?

\[ P(0) \]

\[ P(1) \]

\[ P(2) \]

\[ P(3) \]

\[ P(n) \]

\[ P(n + 1) \]

\[ P(n + 2) \]

\[ P(n + 3) \]

\[ \forall k, P(k) \implies P(k + 1) \]

\[ P(0) \implies P(1) \implies P(2) \implies P(3) \implies \ldots \]

\[ (\forall n \in \mathbb{N}) P(n) \]

Your favorite example of forever...or the natural numbers...


Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.

Check Out: "Four corners".

States connected at a point, can have same color.


Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=0}^{n} i = n(n+1)/2)\) Proof?

Idea: assume predicate \( P(n) \) for \( n = k \), \( P(k) \) is \( \sum_{i=0}^{k} i = \frac{k(k+1)}{2} \).

Is predicate, \( P(n) \) true for \( n = k + 1 \)?

\[
\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}
\]

How about \( k + 2 \). Same argument starting at \( k + 1 \) works!

Induction Step. \( P(k) \implies P(k + 1) \).

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \( P(0) \) is \( \sum_{i=0}^{0} i = 0(0+1)/2 \) Base Case.

Statement is true for \( n = 0 \) \( P(0) \) is true

plus inductive step \( \implies \) true for \( n = 1 \) \( P(0) \implies P(1) \implies P(2) \implies P(3) \)

\[
\vdots \]

true for \( n = k \) \( \implies \) true for \( n = k + 1 \) \( P(k) \implies P(k+1) \implies P(k+2) \)

\[
\vdots \]

Predicate, \( P(n) \), True for all natural numbers! Proof by Induction.


Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

Fact: Swapping red and blue gives another valid colors.
Two color theorem: proof illustration.

Base Case. 1. Add line. 2. Get inherited color for split regions 3. Switch on one side of new line. (Fixes conflicts along line, and makes no new ones.) Algorithm gives $P(k) \implies P(k+1)$.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ has to have one hole.
Proof: The remainder of $2^{2n}$ divided by 3 is 1.
Base case: true for $k = 0$. $2^0 = 1$
Ind Hyp: $2^{2k} = 3a + 1$ for integer $a$.

$2^{2(k+1)} = 2^{2k+2} = 4 \cdot 2^{2k} = 4 \cdot (3a + 1) = 12a + 3 + 1 = 3(4a + 1) + 1$
a integer $\implies (4a + 1) is an integer.

Strenthening Induction Hypothesis.

Theorem: The sum of the first $n$ odd numbers is a perfect square.
Theorem: The sum of the first $n$ odd numbers is $n^2$.
Base Case 1 (first odd number) is $1^2$.
Induction Hypothesis: Sum of first $k$ odds is perfect square $a^2 = k^2$.
Induction Step 1. The $(k + 1)$st odd number is $2k + 1$.
2. Sum of the first $k + 1$ odds is $a^2 + 2k + 1 = k^2 + 2k + 1$
3. $k^2 + 2k + 1 = (k + 1)^2$
... $P(k+1)$!

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.
Proof:
Base case: A single tile works fine.
The hole is adjacent to the center of the $2 \times 2$ square.
Induction Hypothesis: Any $2^n \times 2^n$ square can be tiled with a hole at the center.

$2^{n+1}$

$2^{n+1}$

$2^n$

What to do now???

Tiling Cory Hall Courtyard.

Use these L-tiles.

To tile this $4 \times 4$ courtyard.

Tiled $4 \times 4$ square with $2 \times 2$ L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n with L-tiles (with a hole) for every $n$!

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n to leave a hole adjacent anywhere.
Better theorem: better induction hypothesis!
Base case: Sure. A tile is fine.
Flipping the orientation can leave hole anywhere.
Induction Hypothesis: "Any $2^n \times 2^n$ square can be tiled with a hole anywhere."
Consider $2^{n+1} \times 2^{n+1}$ square.

Use induction hypothesis in each.

Use L-tile and ... we are done.
Well Ordering Principle and Induction.

If \(\forall n)P(n)\) is not true, then \(\exists n\) P(n).
Consider smallest \(m\) with \(\neg P(m)\), \(m \geq 0\).

\(P(m - 1) \rightarrow P(m)\) must be false (assuming \(P(0)\) holds.)
This is a proof of the induction principle!
I.e.,
\[
(\forall n)P(n) \rightarrow ((\forall n)\neg P(n - 1) \rightarrow P(n)).
\]

Well ordering principle.
Thm: All natural numbers are interesting.
0 is interesting...
Let \(n\) be the first uninteresting number.
But \(n - 1\) is interesting and \(n\) is uninteresting,
so this is the first uninteresting number.
But this is interesting.
Thus, there is no smallest uninteresting natural number.
Thus: All natural numbers are interesting.

Tournaments have short cycles

Def: A round robin tournament on \(n\) players: every player \(p\) plays every other player \(q\), and either \(p \rightarrow q\) (\(p\) beats \(q\)) or \(q \rightarrow p\) (\(q\) beats \(p\)).

Def: A cycle: a sequence of \(p_1, \ldots, p_k\), \(p_1 \rightarrow p_2, \ldots, p_k \rightarrow p_1\).

Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournament has a cycle of length 3 if at all.

Assume the the smallest cycle is of length \(k\).
Case 1: Of length 3. Done.
Case 2: Of length larger than 3.

Well ordering principle.

Thm: All natural numbers are interesting.
0 is interesting...
Let \(n\) be the first uninteresting number.
But \(n - 1\) is interesting and \(n\) is uninteresting,
so this is the first uninteresting number.
But this is interesting.
Thus, there is no smallest uninteresting natural number.
Thus: All natural numbers are interesting.

Tournaments have long paths.

Def: A round robin tournament on \(n\) players: all pairs \(p\) and \(q\) play, and either \(p \rightarrow q\) (\(p\) beats \(q\)) or \(q \rightarrow p\) (\(q\) beats \(p\)).

Def: A Hamiltonian path: a sequence
\(p_1, \ldots, p_n\) (\(\forall 0 \leq i < n\)) \(p_i \rightarrow p_{i+1}\).

Base: True for two vertices.
(Also for one, but two is more fun as base case!)
Tournament on \(n = 1\) people.
Remove arbitrary person \(\rightarrow \) yield tournament on \(n - 1\) people.

By induction hypothesis: There is a sequence \(p_1, \ldots, p_n\)
contains all the people where \(p_1 \rightarrow p_{i+1}\).

If \(p\) is big winner, put at beginning. Big loser at end.
If neither, find first place \(i\), where \(p\) beats \(p_i\).
\(p_1, \ldots, p_i, p, p_{i+1}, \ldots, p_n\) is Hamiltonian path.
Horses of the same color...

**Theorem:** All horses have the same color.

- **Base Case:** $P(1)$ - trivially true.
- **New Base Case:** There are two horses of the same color.

Induction Hypothesis: $P(k)$ - Any $k$ horses have the same color.

Induction step $P(k+1)$?

- First $k$ have same color by $P(k)$. 1, 2, 3, ..., $k+1$
- Second $k$ have same color by $P(k)$. 1, 2, 3, ..., $k+1$
- A horse in the middle in common! 1, 2, 3, ..., $k+1$
- All k must have the same color!! 1, 2, 3, ..., $k+1$

How about $P(1) \implies P(2)$?

Fix base case.

There are two horses of the same color. ...Still doesn’t work!!

(There are two horses is $\not\equiv$ For all two horses!!)

Of course it doesn’t work.

As we will see, it is more subtle to catch errors in proofs of correct theorems!!

---

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must commit ritual suicide that day.

No islander knows there own eye color, but knows everyone else's.

All islanders have green eyes!

First rule of island: Don’t talk about eye color!

Visitor: "I see someone has green eyes."

Result: On day 100, they all do the ritual.

Why?

---

Summary: principle of induction.

Today: More induction.

($P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for $n$ starting from $n_0$

Base Case: Prove $P(n_0)$.

Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n + 1)$.

Statement is proven!

**Strong Induction:**

($P(0) \land ((\forall n \in N)(P(n) \implies P(n + 1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

**Strengthen theorem statement.**

Sum of first $n$ odds is $n^2$.

Hole anywhere.

**Not same as strong induction.** E.g., used in product of primes proof.

Induction = Recursion.

---

They know induction.

Thm: If there are $n$ villagers with green eyes they do ritual on day $n$.

**Proof:**

Base: $n = 1$. Person with green eyes does ritual on day 1.

Induction hypothesis:

If there were $n$ people with green eyes, they would do ritual on day $n$.

Induction step:

On day $n + 1$, a green eyed person sees $n$ people with green eyes. But they didn’t do the ritual.

So there must be $n + 1$ people with green eyes.

One of them, is me.

Sad.

Wait! Visitor added no information.

Summary: principle of induction.

($P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n))$

Variations:

Strong Induction:

($P(0) \land ((\forall n \in N)(P(n) \implies P(n + 1)))) \implies (\forall n \in N)(P(n))$

Different Starting Point:

($P(1) \land ((\forall n \in N)((n \geq 1) \implies P(n)) \implies P(n + 1))))$$

Statement to prove: $P(n)$ for $n$ starting from $n_0$

Base Case: Prove $P(n_0)$.

Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n + 1)$.

Statement is proven!