Lecture 15: More Probability.

Events, Conditional Probability, Independence, Bayes' Rule

Summary.

Modeling Uncertainty: Probability Space

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4. Event: "subset of outcomes." $A \subseteq \Omega$. $Pr[A] = \sum_{w \in A} Pr[\omega]$
5. Some calculations.

CS70: Onwards.

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Events, Conditional Probability, Independence, Bayes' Rule

1. Probability Basics Review
2. Events
3. Conditional Probability
4. Independence of Events
5. Bayes' Rule

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$\implies$ not surprising to have something like $n + \sqrt{\pi n}/2$ heads

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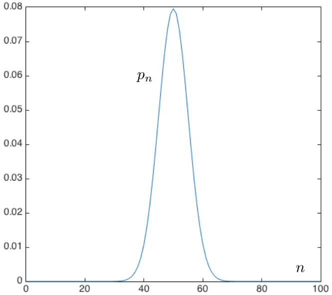

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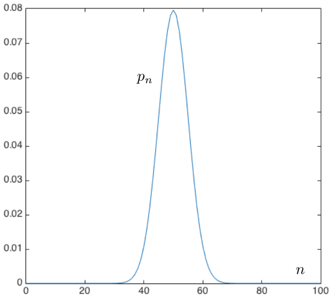

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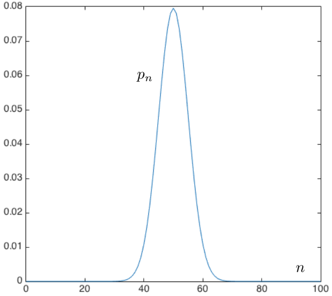

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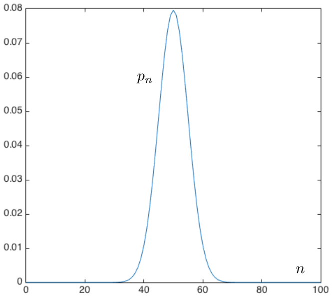

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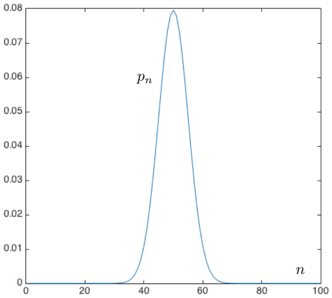

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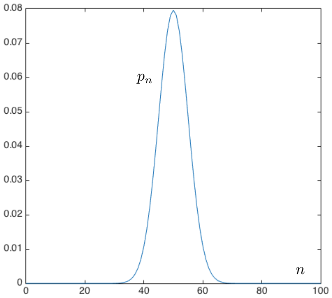

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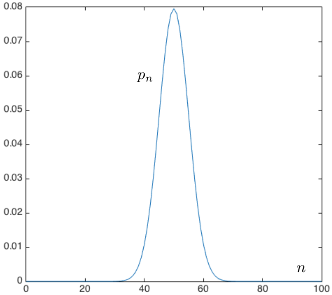

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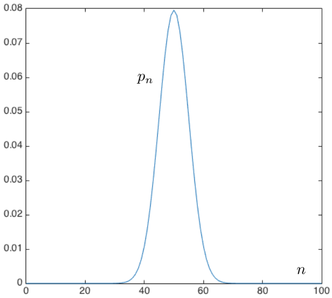

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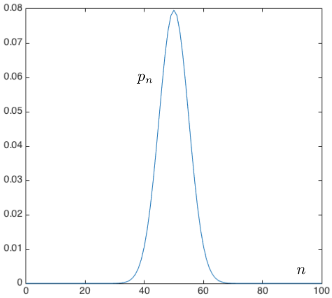

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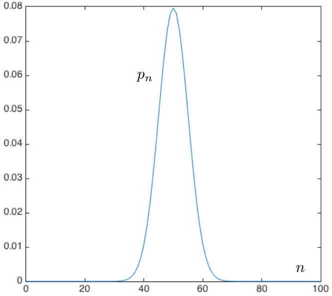

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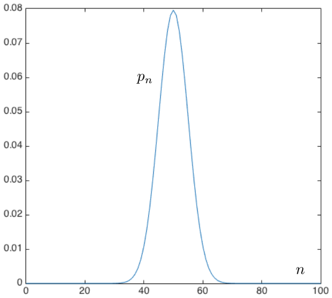

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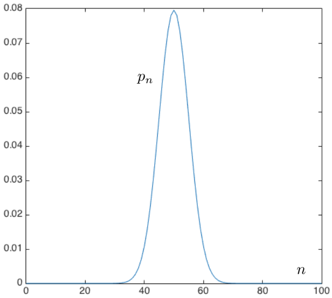

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Obvious. Straightforward. Use definition of probability of events.

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(a) $Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$;

(inclusion-exclusion property)

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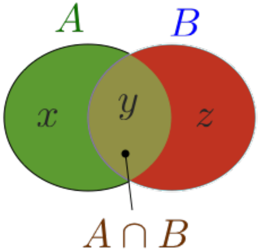

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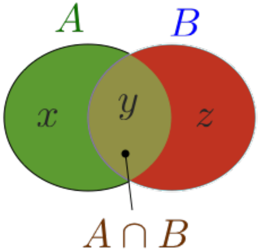

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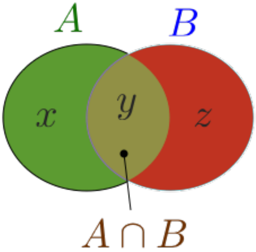

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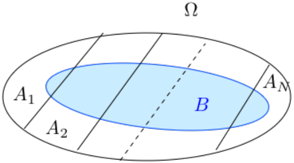
Another view. Any $\omega \in A \cup B$ is in $A \cap \overline{B}$, $A \cup B$, or $\overline{A} \cap B$. So, add it up.

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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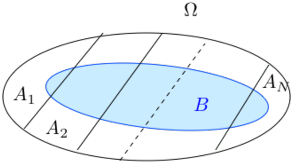

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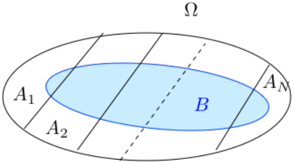

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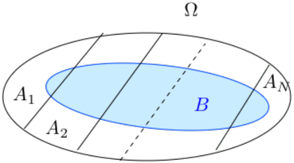

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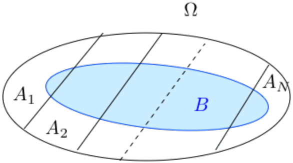

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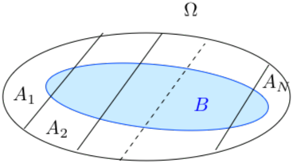

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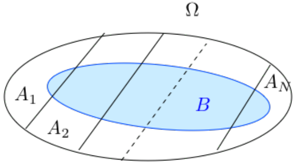

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Add it up.

Roll a Red and a Blue Die.

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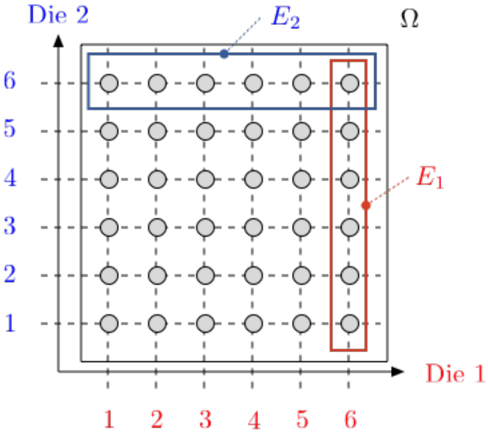

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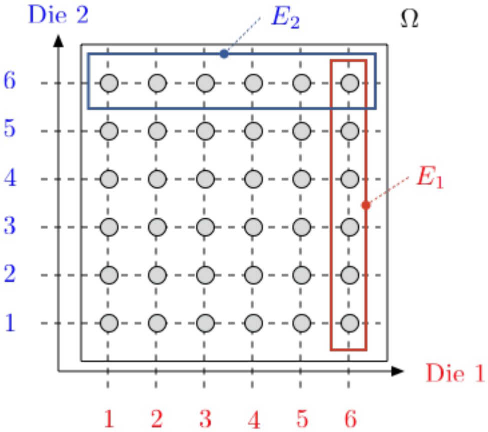

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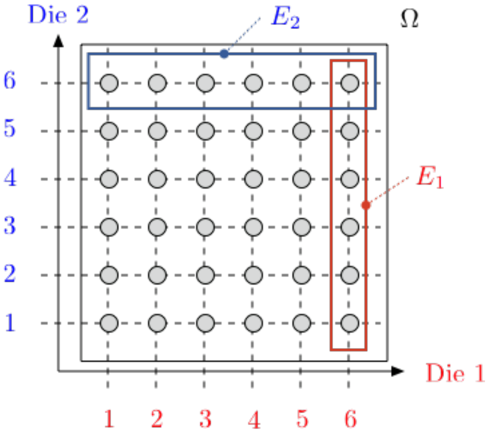
E_1 = 'Red die shows 6';

Roll a Red and a Blue Die.



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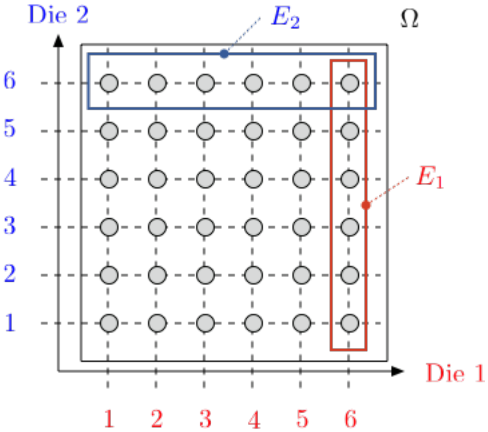

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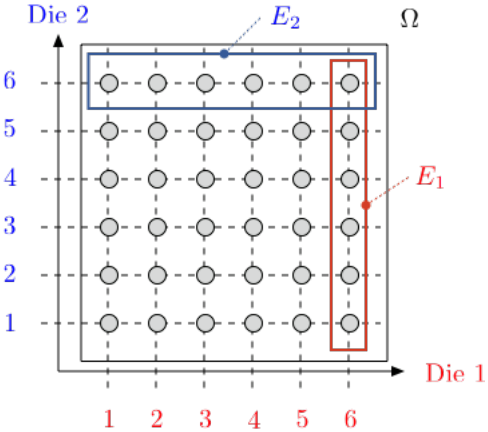

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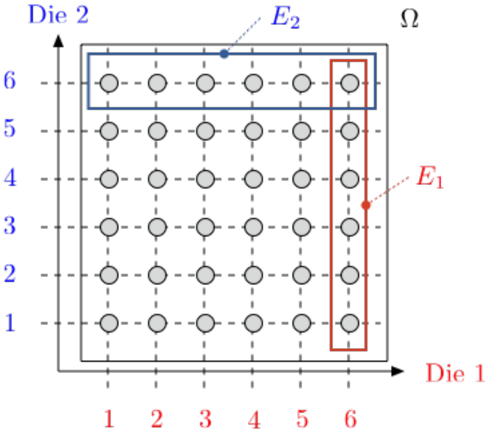

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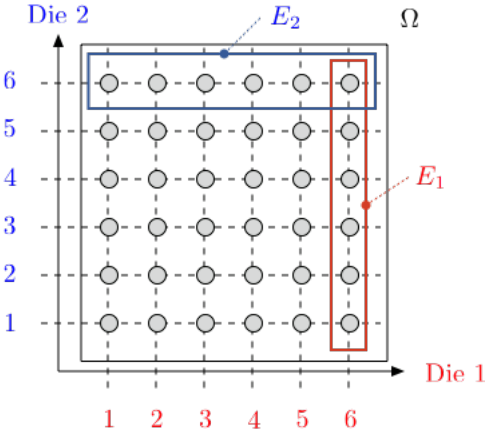

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$Pr[E_1] = \frac{6}{36}, Pr[E_2] = \frac{6}{36}, Pr[E_1 \cup E_2] = \frac{11}{36}.$

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Two coin flips.

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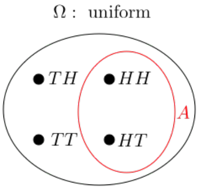

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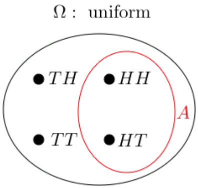

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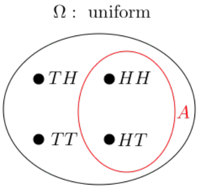

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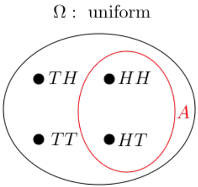

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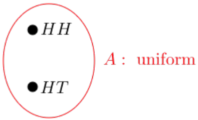

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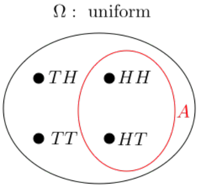

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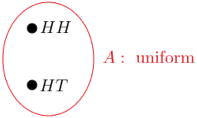

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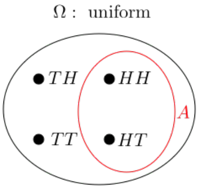

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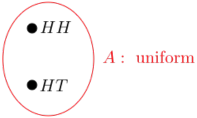

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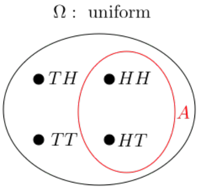

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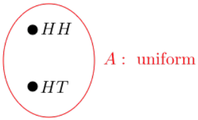

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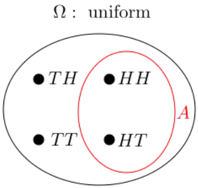

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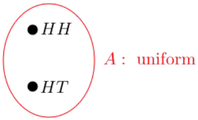

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A similar example.

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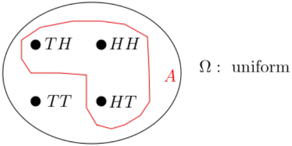

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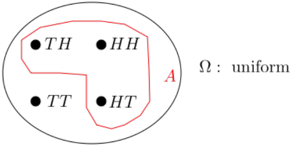

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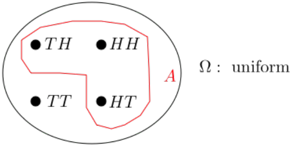

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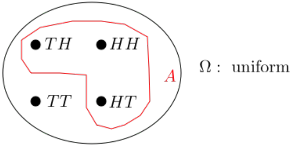

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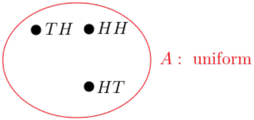

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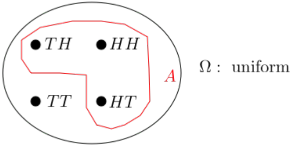

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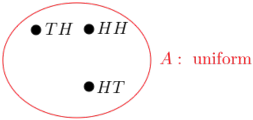

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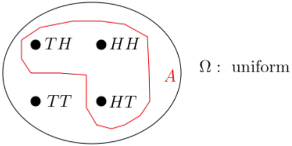

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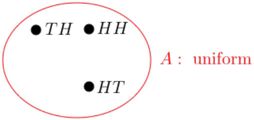

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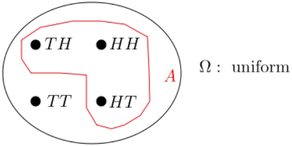

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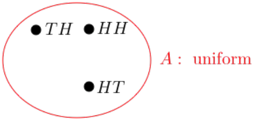

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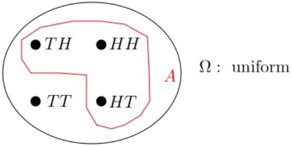

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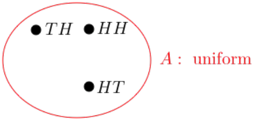

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Conditional Probability: A non-uniform example

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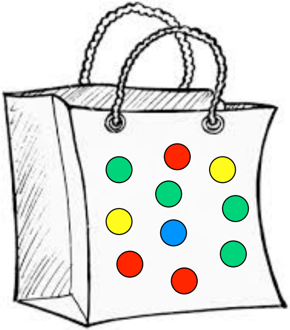
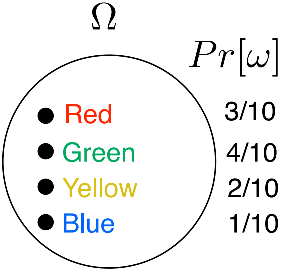


Physical experiment



Probability model

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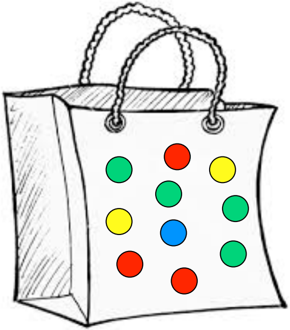

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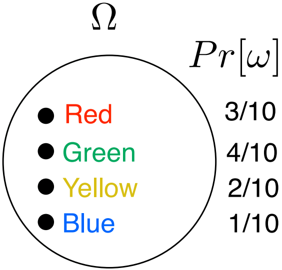

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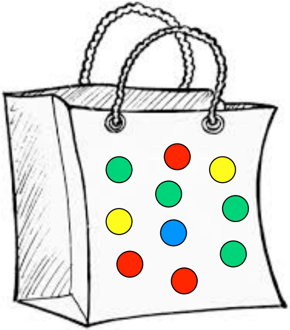

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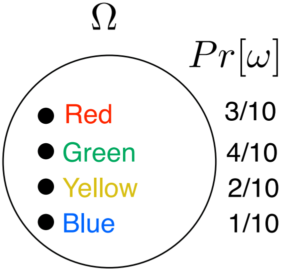

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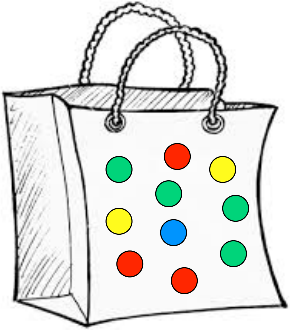

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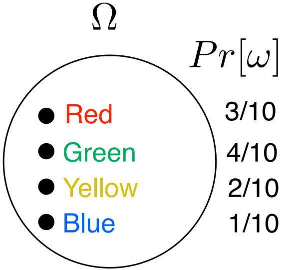

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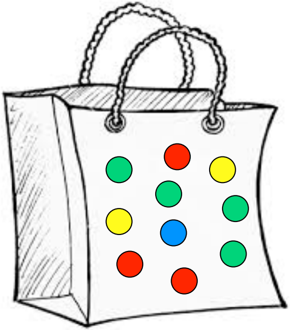

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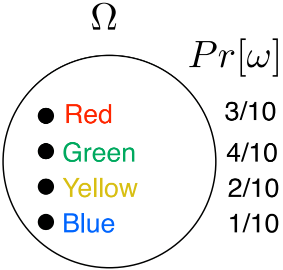

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Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

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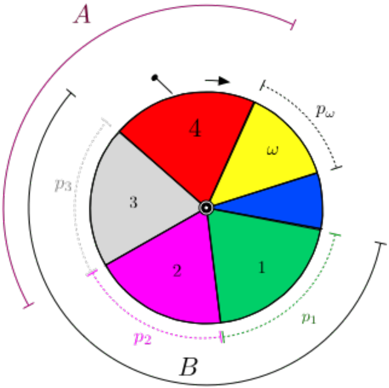

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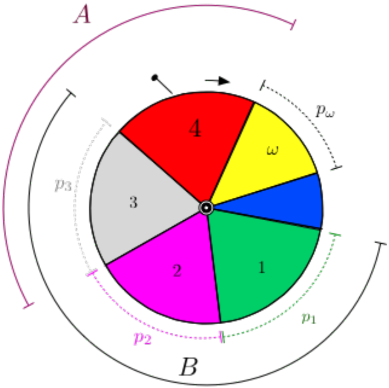

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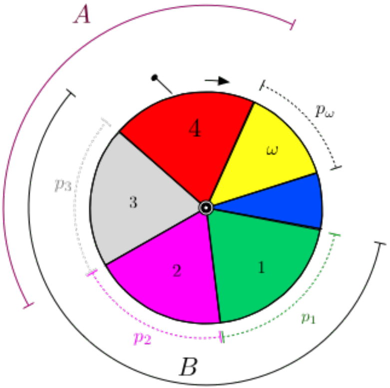

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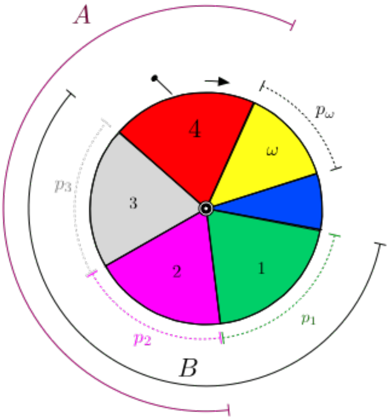

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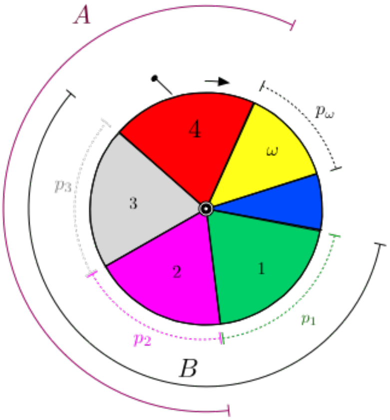

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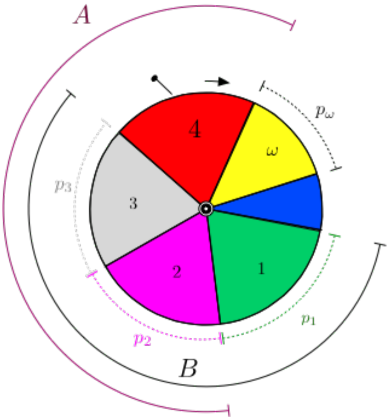

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Definition: The **conditional probability** of B given A is

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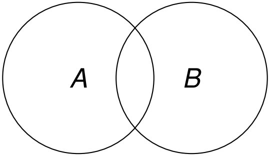

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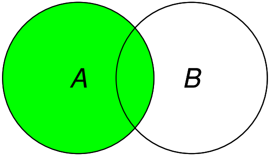

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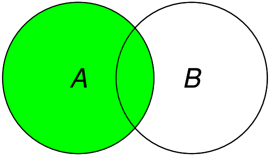

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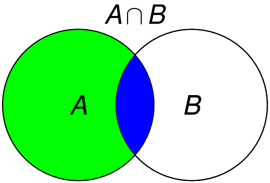

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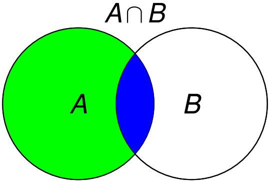

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More fun with conditional probability.

Toss a red and a blue die, sum is 4,

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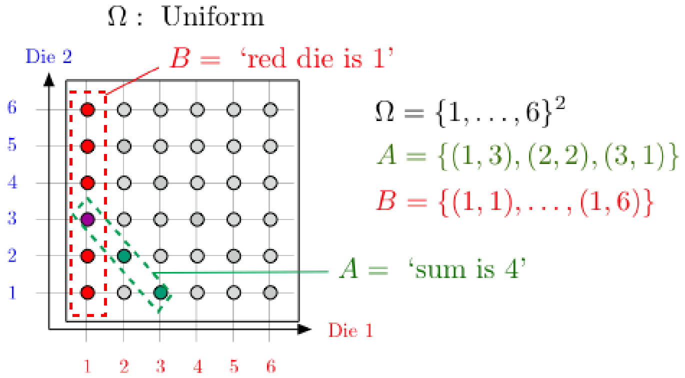

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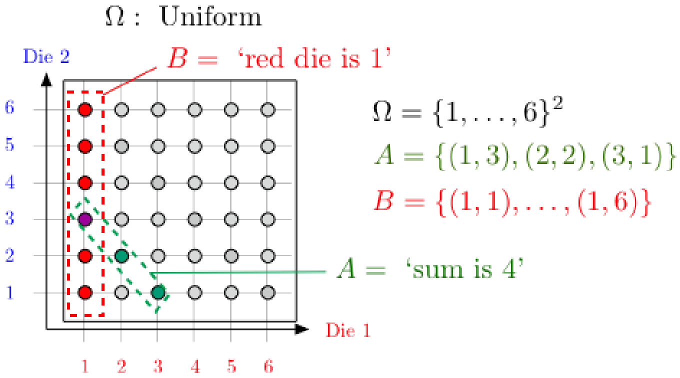

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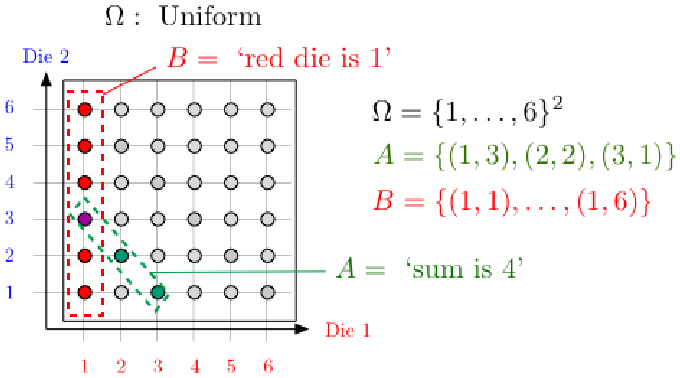

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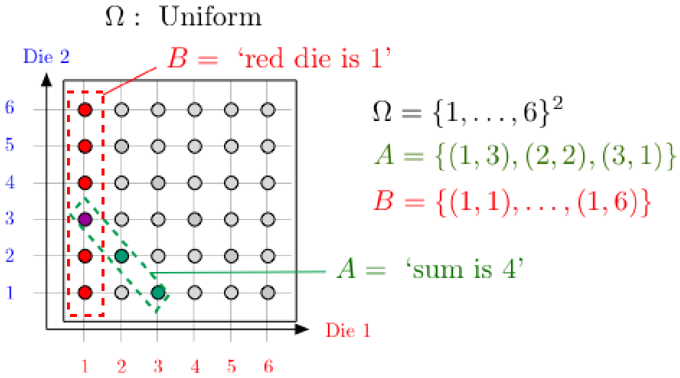

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B is more likely given *A*.

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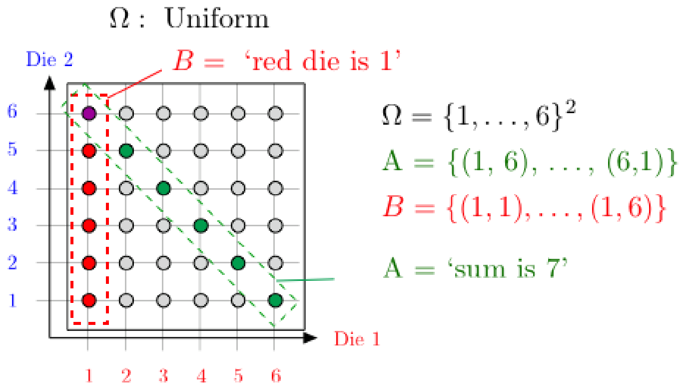

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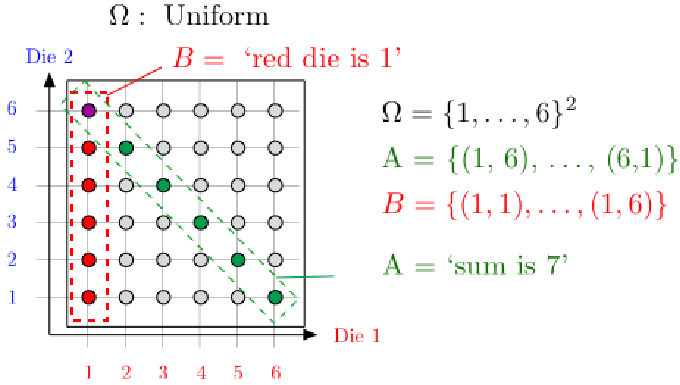

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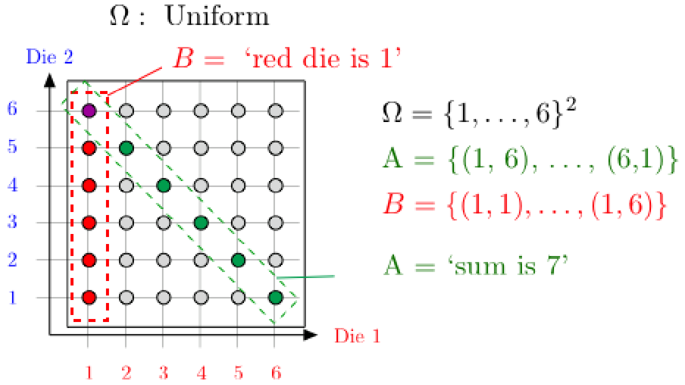

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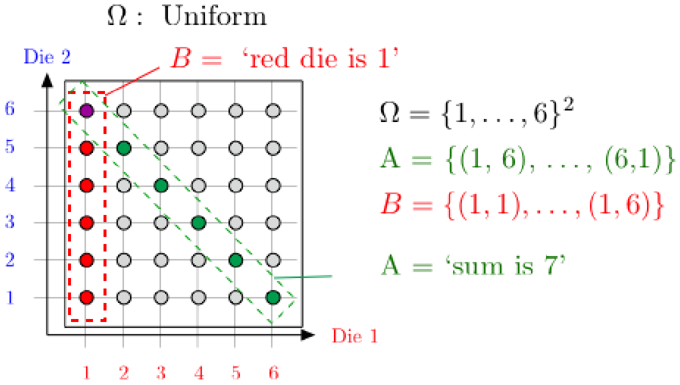

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Observing *A* does not change your mind about the likelihood of *B*.

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Suppose I toss 3 balls into 3 bins.

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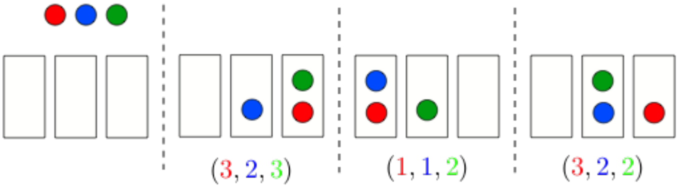
$$\Omega = \{1, 2, 3\}^3$$



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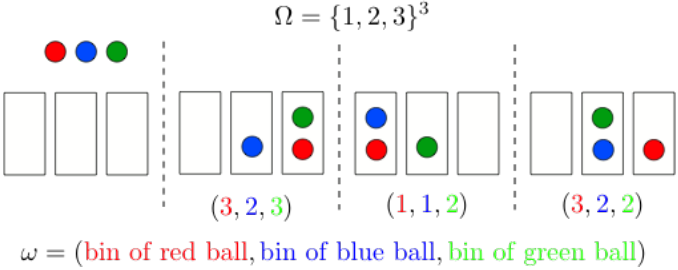

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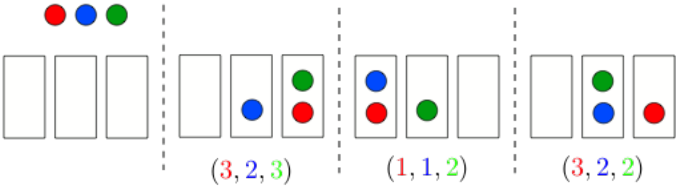

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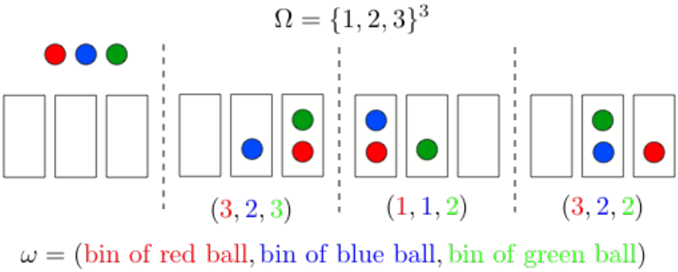

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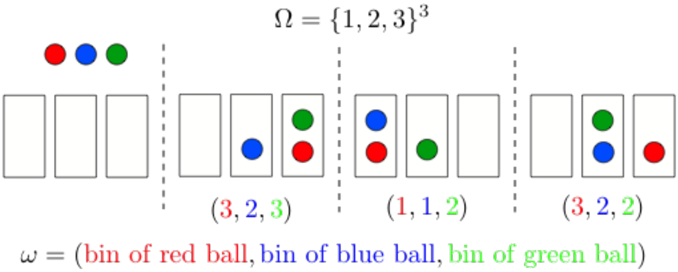

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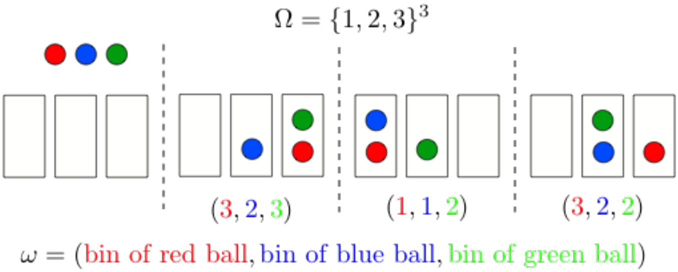

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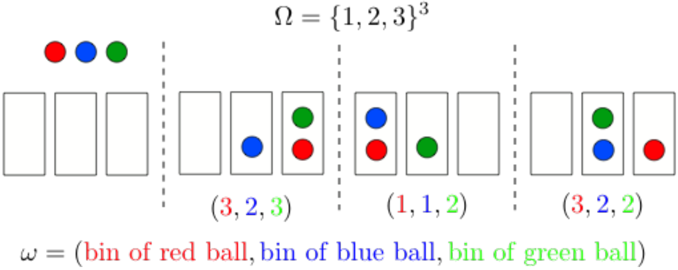

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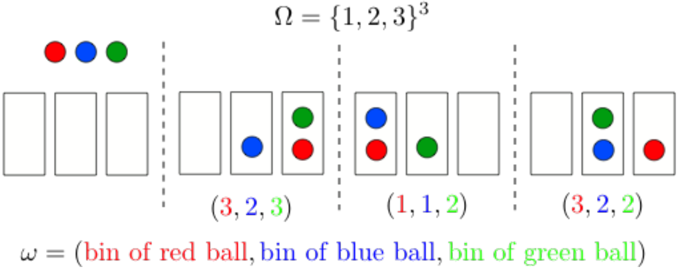

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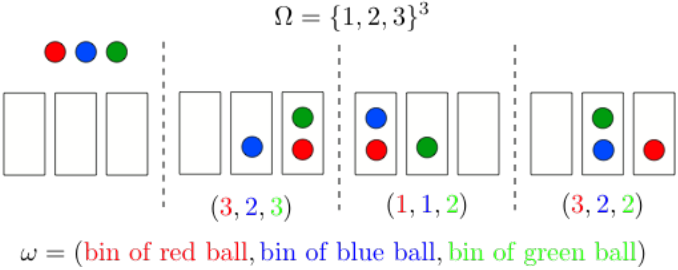

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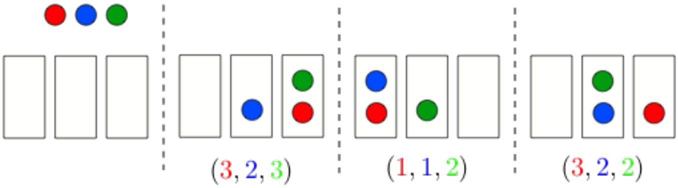

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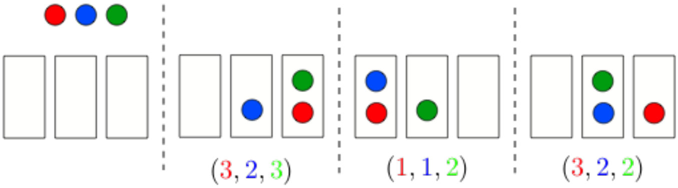

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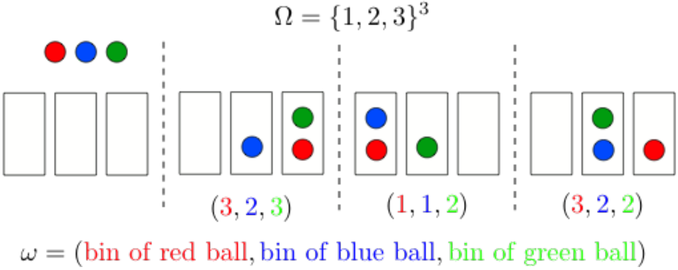

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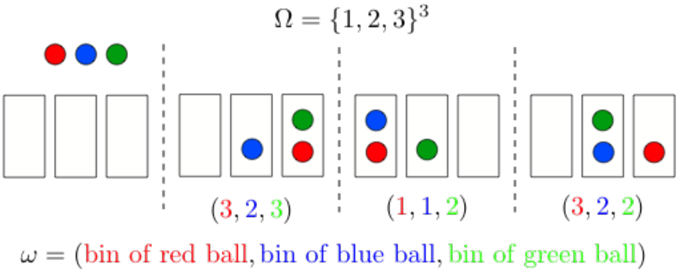

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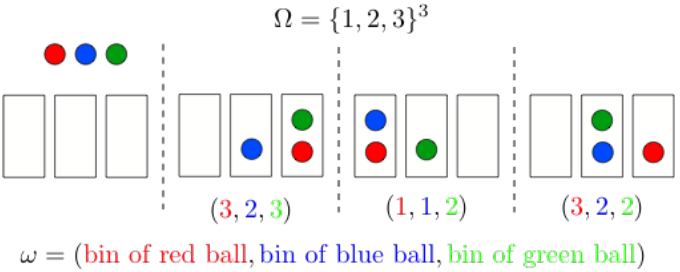

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Second bin is empty $\implies$ first bin is more likely to contain ball(s).

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The likelihood of 51st heads does not depend on the previous flips.

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Thus, the result holds for $n+1$. $\square$

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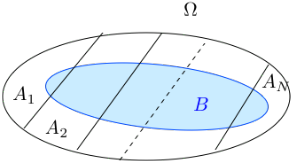
More about such questions later. For fun, check "N. Taleb: Fooled by randomness."

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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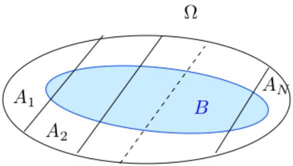

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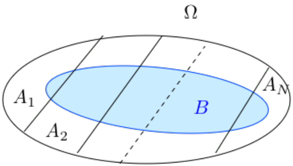

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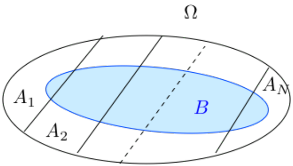

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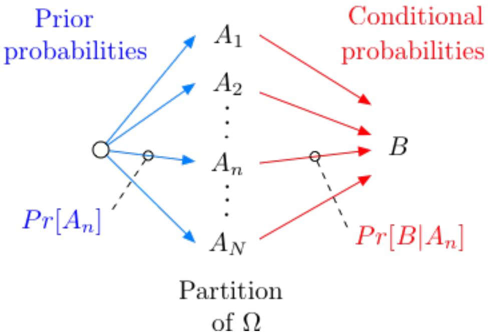

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Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$

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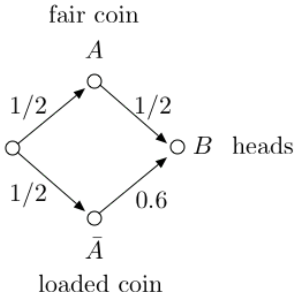

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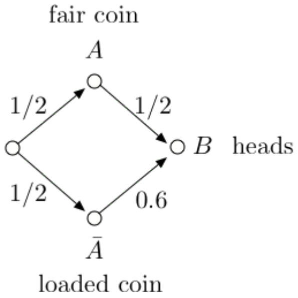

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$m := 100(1/2)(1/2)$ are such that A and B occur and
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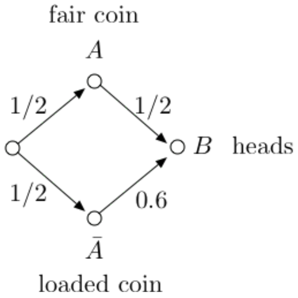

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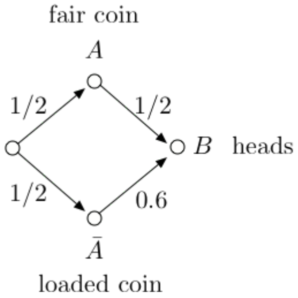

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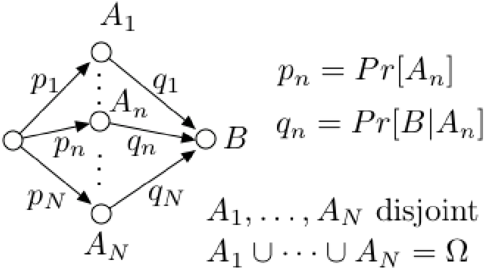

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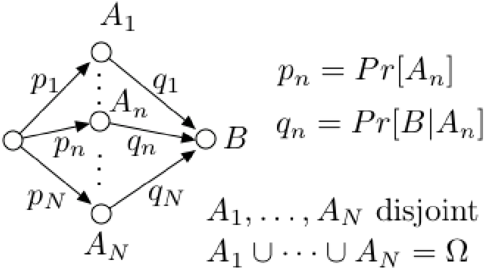

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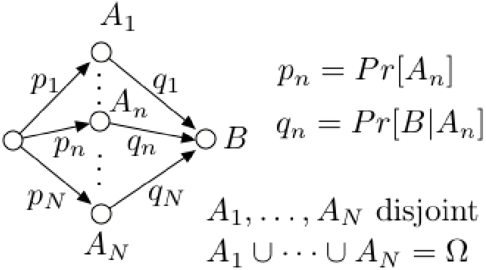

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Hence,

$$Pr[A_n|B] = \frac{p_n q_n}{\sum_m p_m q_m}.$$

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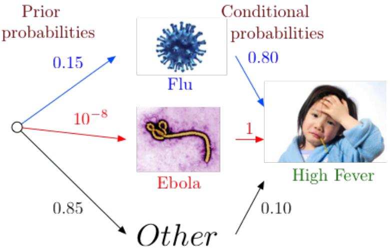

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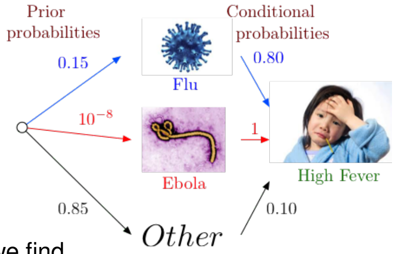

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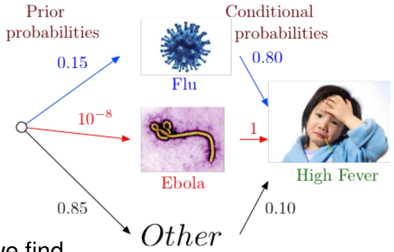

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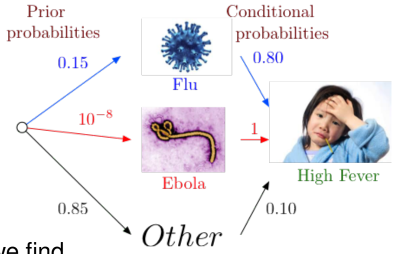

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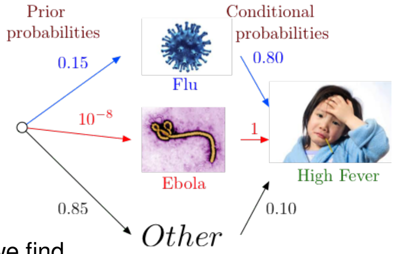

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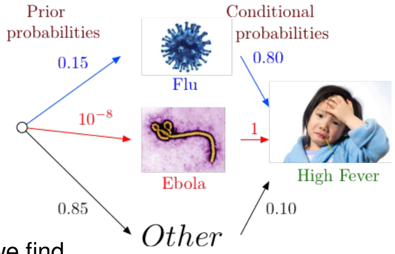

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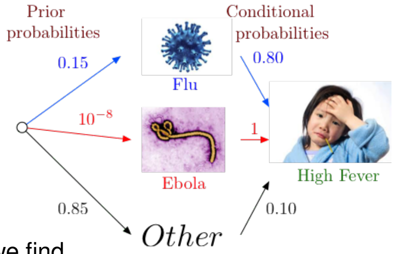

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Bayes' Rule Operations

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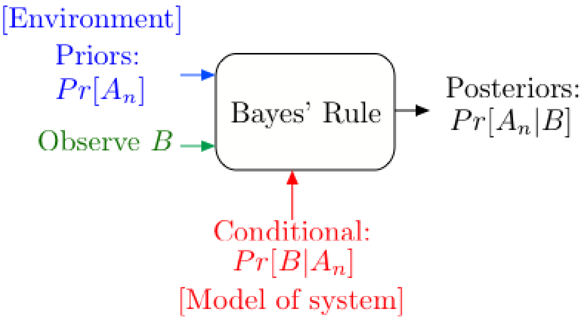

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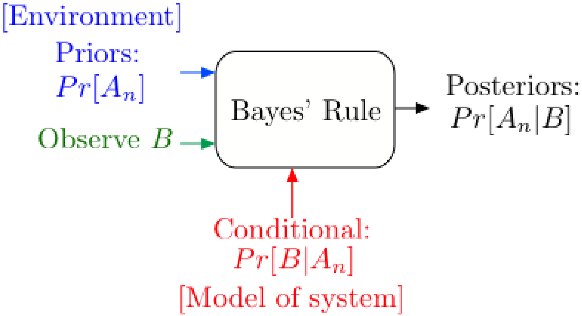

Bayes' Rule is the canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A

FIG. 3. *Joshua Bayes (1671–1746).*

$Pr[B|A]$ →

Max over B

B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Testing for disease.

Let’s watch TV!!

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$$Pr[A|B]???$$

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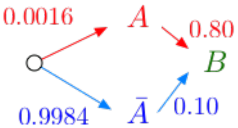

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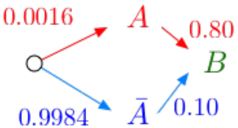

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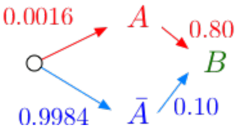

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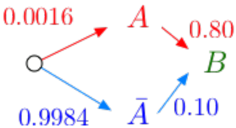

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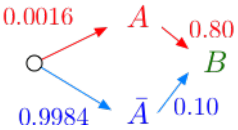

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A 1.3% chance of prostate cancer with a positive PSA test.

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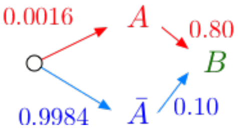

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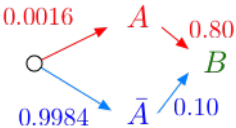

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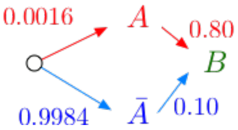

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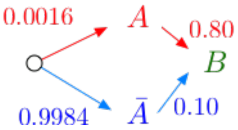

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- All these are possible:

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