Refresh: Counting.

First Rule of counting: Objects from a sequence of choices:

\[ n_1 \times n_2 \times \cdots \times n_k \text{ objects.} \]

Second Rule of counting: If order does not matter.

Count with order: Divide by number of orderings/sorted object.

Typically: \( \binom{n}{k} \).

Stars and Bars: Sample \( k \) objects with replacement from \( n \).

Order doesn’t matter. \( k \) stars \( n-1 \) bars. Typically: \( \binom{n+k-1}{k} \) or \( \binom{n+k-1}{n-1} \).

Inclusion/Exclusion: two sets of objects.

Add number of each and then subtract intersection of sets.

Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.

Pascal’s Triangle Example: \( \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \).

RHS: Number of subsets of \( n+1 \) items size \( k \).

LHS: \( \binom{n}{k} \) counts subsets of \( n+1 \) items with first item.

\( \binom{k}{1} \) counts subsets of \( n+1 \) items without first item.

Disjoint – so add!

CS70: On to probability.

Key Points

- Uncertainty does not mean “nothing is known”
- How to best make decisions under uncertainty?
  - Buy stocks
  - Detect signals (transmitted bits, speech, images, radar, diseases, etc.)
  - Control systems (Internet, airplane, robots, self-driving cars, schedule surgeries in a hospital, etc.)
- How to best use ‘artificial’ uncertainty?
  - Play games of chance
  - Design randomized algorithms.
- Probability
  - Models knowledge about uncertainty
  - Optimizes use of knowledge to make decisions

Random Experiment: Flip one Fair Coin

Flip a fair coin: (One flips or tosses a coin)

- Possible outcomes: Heads (H) and Tails (T)
  (One flip yields either ‘heads’ or ‘tails’.)
- Likelihoods: \( H : 50\% \) and \( T : 50\% \)

Uncertainty: vague, fuzzy, confusing, scary, hard to think about.
Probability: A precise, unambiguous, simple(!) way to think about uncertainty.

Our mission: help you discover the serenity of Probability, i.e., enable you to think clearly about uncertainty.

Your cost: focused attention and practice on examples and problems.
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
  - A set $\Omega$ of outcomes: $\Omega = \{H, T\}$.
  - A probability assigned to each outcome: $Pr[H] = 0.5$, $Pr[T] = 0.5$.

Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

- Possible outcomes: Heads ($H$) and Tails ($T$)
- Likelihoods: $H : p \in (0,1)$ and $T : 1 - p$
- Frequentist Interpretation:
  - Flip many times $\Rightarrow$ Fraction $1 - p$ of tails
- Question: How can one figure out $p$? Flip many times
- Tautology? No: Statistical regularity!

Random Experiment: Flip two Fair Coins

Flips two coins attached by a spring:

- Possible outcomes: $\{HH, HT, TH, TT\}$
- Likelihoods: $HH : 0.4$, $HT : 0.1$, $TH : 0.1$, $TT : 0.4$
- Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.

Random Experiment: Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: $\{HT, TH\}$
- Likelihoods: $HT : 0.5$, $TH : 0.5$
- Note: Coins are glued so that they show different faces.
Flipping Two Coins

Here is a way to summarize the four random experiments:

- $\Omega$ is the set of possible outcomes;
- Each outcome has a probability (likelihood);
- The probabilities are $\geq 0$ and add up to 1;
- Fair coins: [1]; Glued coins: [3, [4];
- Spring-attached coins: [2];

Important remarks:
- Each outcome describes the two coins.
- E.g., $HT$ is one outcome of each of the above experiments.
- It is wrong to think that the outcomes are $\{H,T\}$ and that one picks twice from that set.
- Indeed, this viewpoint misses the relationship between the two flips.
- Each $\omega \in \Omega$ describes one outcome of the complete experiment.
- $\Omega$ and the probabilities specify the random experiment.

Probability Space.

1. A "random experiment":
   - (a) Flip a biased coin;
   - (b) Flip two fair coins;
   - (c) Deal a poker hand.
   - Thus, $2^n$ possible outcomes.
   - Note: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\} = (H,T)^n$.
   - $A^n := \{(a_1, \ldots, a_n) | a_i \in A_i, \ldots, a_n \in A_n\}$. $|A^n| = |A|^n$.
   - Likelihoods: $1/2^n$ each.

Flipping $n$ times

Flip a fair coin $n$ times (some $n \geq 1$):
- Possible outcomes: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\}$.
- $\Omega = \{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\}$.
- $|\Omega| = 2^n$.
- $A^n := \{(a_1, \ldots, a_n) | a_i \in A_i, \ldots, a_n \in A_n\}$. $|A^n| = |A|^n$.
- Likelihoods: $1/2^n$ each.

Roll two Dice

Roll a balanced 6-sided die twice:
- Possible outcomes: $\{(a,b) | 1 \leq a, b \leq 6\}$.
- Likelihoods: $1/36$ for each.
Probability Space: Formalism.

In a uniform probability space each outcome \( \omega \) is equally probable:

\[ \Pr[\omega] = \frac{1}{|\Omega|} \text{ for all } \omega \in \Omega. \]

Examples:
- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
- Flipping a biased coin is not a uniform probability space.

Probability Space: Formalism

Simplest physical model of a uniform probability space:

\begin{itemize}
  \item Red
  \item Green
  \item Maroon
\end{itemize}

\[ \frac{1}{8} \]

\begin{itemize}
  \item Physical experiment
  \item Probability model
\end{itemize}

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]

\[ \Pr[\text{Red}] = \frac{3}{10}, \Pr[\text{Green}] = \frac{4}{10}, \text{ etc.} \]

Note: Probabilities are restricted to rational numbers: \( \frac{N}{k} \).

An important remark

- The random experiment selects one and only one outcome in \( \Omega \).
- For instance, when we flip a fair coin twice
  - \( \Omega = \{ HH, TH, HT, TT \} \)
  - The experiment selects one of the elements of \( \Omega \).
- In this case, its wrong to think that \( \Omega = \{ H, T \} \) and that the experiment selects two outcomes.
- Why? Because this would not describe how the two coin flips are related to each other.
- For instance, say we glue the coins side-by-side so that they face up the same way. Then one gets \( HH \) or \( TT \) with probability 50% each. This is not captured by ‘picking two outcomes.’

Summary of Probability Basics

Modeling Uncertainty: Probability Space

1. Random Experiment
2. Probability Space: \( \Pr[\omega] \in [0,1] \); \( \sum \Pr[\omega] = 1 \).
3. Uniform Probability Space: \( \Pr[\omega] = 1/|\Omega| \) for all \( \omega \in \Omega \).
Onwards in Probability.

Events, Conditional Probability, Independence, Bayes' Rule

Onwards in Probability.

Events, Conditional Probability, Independence, Bayes' Rule

CS70: On to Events.

Today: Events.

Probability Basics Review

Setup:

▶ Random Experiment. Flip a fair coin twice.
▶ Probability Space.
▶ Sample Space: \(\Omega\) 
\(\Omega = \{HH, HT, TH, TT\}\) (Note: Not \(\Omega = \{H, T\}\) with two picks!)
▶ Probability: \(\Pr[\omega]\) for all \(\omega \in \Omega\).
\(\Pr[HH] = \cdots = \Pr[TT] = \frac{1}{4}\)
1. \(0 \leq \Pr[\omega] \leq 1\).
2. \(\sum_{\omega \in \Omega} \Pr[\omega] = 1\).

Probability of exactly one 'heads' in two coin flips?

Idea: Sum the probabilities of all the different outcomes that have exactly one 'heads': \(HT, TH\).

This leads to a definition!
Definition:

▶ An event, \(E\), is a subset of outcomes: \(E \subset \Omega\).
▶ The probability of \(E\) is defined as \(\Pr[E] = \sum_{\omega \in E} \Pr[\omega]\).

Event: Example

\(\Omega = \{\text{Red, Green, Yellow, Blue}\}\)
\(\Pr[\text{Red}] = \frac{3}{10}\), \(\Pr[\text{Green}] = \frac{4}{10}\), etc.

\(E = \{\text{Red, Green}\} \Rightarrow \Pr[E] = \frac{3}{10} + \frac{4}{10} = \Pr[\text{Red}] + \Pr[\text{Green}]\).
Probability of exactly one heads in two coin flips?

Sample Space, $\Omega = \{HH, HT, TH, TT\}$. Uniform probability space: $Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$.

Event, $E$, "exactly one heads": $\{TH, HT\}$.

Probability of $n$ heads in 100 coin tosses.

Sample space: $\Omega = \{H, T\}^{100}$; $|\Omega| = 2^{100}$.

Event $E_n = \text{\# heads}$; $|E_n| = \binom{100}{n}$.

Uniform probability space: $Pr[\omega] = \frac{1}{|\Omega|}$.

Stirling formula (for large $n$):

$\approx \sqrt{\frac{2\pi n}{n/e}} n^{n} e^{-n} = \sqrt{\frac{2\pi n}{n/e}} n^{n} e^{-n}$.

Calculation.

Example: 20 coin tosses.

Sample space: $\Omega = \text{set of 20 fair coin tosses}$.

$\Omega = \{\ T, H \}^{20} = (0,1)^{20}; \ |\Omega| = 2^{20}$.

Roll a red and a blue die.

20 coin tosses

What is more likely?

- $\omega_1 := (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$, or
- $\omega_2 := (1,0,1,1,0,0,0,0,1,0,1,0,1,0,1,0,1,1,0,0,0,0)$?

Answer: Both are equally likely: $Pr[\omega_1] = Pr[\omega_2] = \frac{1}{2}^{20}$.

What is more likely?

- $\{E_1\} \text{ Twenty Hs out of twenty, or}$
- $\{E_2\} \text{ Ten Hs out of twenty,}$

Answer: Ten Hs out of twenty.

Why? There are many sequences of 20 tosses with ten Hs; only one with twenty.

$\Rightarrow Pr[E_1] = \frac{C_{10}}{C_{20}} \ll Pr[E_2] = \frac{C_{10}}{C_{20}}$.

$|E_2| = \frac{2^{20}}{10} 
\approx 184.756$.

Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega = \text{set of 100 coin tosses} = \{H, T\}^{100}$.

$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

$|E| = \binom{100}{50}$.

Choose 50 positions out of 100 to be heads.

$|E| = \binom{100}{50}$.

$Pr[E] = \frac{\binom{100}{50}}{2^{100}}$.
Exactly 50 heads in 100 coin tosses.

Summary.

1. Random Experiment
2. Probability Space: $\Omega; Pr[\omega] \in [0,1]; \sum_{\omega} Pr[\omega] = 1.$
3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega.$
4. Event: “subset of outcomes.” $A \subseteq \Omega.$ $Pr[A] = \sum_{\omega \in A} Pr[\omega]$
5. Some calculations.