Today.

Today.

Farewell to modular arithmetic.

Today.

Farewell to modular arithmetic. Until the midterm.

Today.

Farewell to modular arithmetic. Until the midterm. And final.

Today.

Farewell to modular arithmetic. Until the midterm. And final.

Coutability and Uncountability.

Today.

Farewell to modular arithmetic. Until the midterm. And final.

Coutability and Uncountability.

Undecidability.

Farewell (for now) to modular arithmetic...

Modular arithmetic modulo a prime.

Farewell (for now) to modular arithmetic...

Modular arithmetic modulo a prime.

Add, subtract, commutative, associative,

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Modular arithmetic modulo a prime.

Add, subtract, commutative, associative, inverses!

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Allow for solving linear systems, discussing polynomials...

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Why not modular arithmetic all the time?

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$4 > 3$?

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Another problem.

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4 is close to 3.

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4 is close to 3.
But can you get closer?

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Another problem.

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But can you get closer? Sure.

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Modular arithmetic modulo a prime.

Add, subtract, commutative, associative, inverses!
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Another problem.

4 is close to 3.
But can you get closer? Sure. 3.5.

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Modular arithmetic modulo a prime.

Add, subtract, commutative, associative, inverses!
Allow for solving linear systems, discussing polynomials...

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Another problem.

4 is close to 3.
But can you get closer? Sure. 3.5. Closer.

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4 is close to 3.
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For reals numbers we have the notion of limit, continuity, and derivative.......

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For reals numbers we have the notion of limit, continuity, and derivative.......

....and Calculus.

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For reals numbers we have the notion of limit, continuity, and derivative.......

....and Calculus.

For modular arithmetic...no Calculus. Sad face!

Next up: how big is infinity.

Next up: how big is infinity.

- Countable
- Countably infinite.
- Enumeration

How big are the reals or the integers?

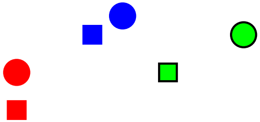
Infinite!

How big are the reals or the integers?

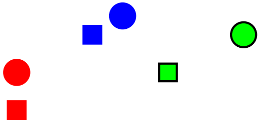
Infinite!

Is one bigger or smaller?

Same size?



Same size?



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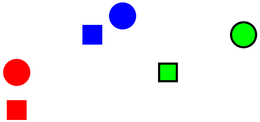
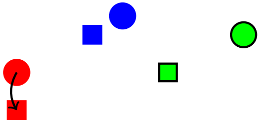
Same size?



Same number?

Make a function f : Circles $\to$ Squares.

Same size?

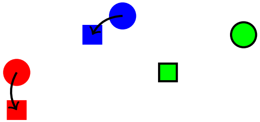


Same number?

Make a function f : Circles $\rightarrow$ Squares.

f(red circle) $=$ red square

Same size?



Same number?

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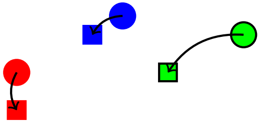
f(red circle) $=$ red square

f(blue circle) $=$ blue square

Same size?



Same number?

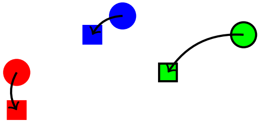
Make a function f : Circles $\to$ Squares.

f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

Same size?



Same number?

Make a function f : Circles $\rightarrow$ Squares.

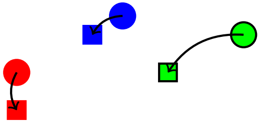
f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

One to one.

Same size?



Same number?

Make a function f : Circles $\rightarrow$ Squares.

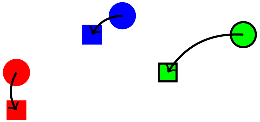
f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

One to one. Each circle mapped to different square.

Same size?



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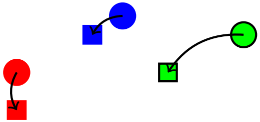
One to one. Each circle mapped to different square.

One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

Same size?



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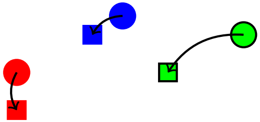
Onto.

Same size?



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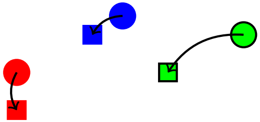
Onto. Each square mapped to from some circle .

Same size?



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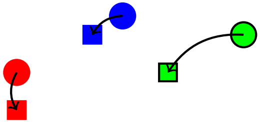
Onto. Each square mapped to from some circle .

Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

Same size?



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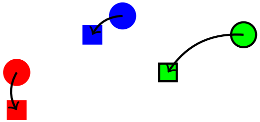

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One to one. Each circle mapped to different square.
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Onto. Each square mapped to from some circle .
Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

Isomorphism principle: If there is $f : D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.
One to One:

Isomorphism principle.

Given a function, $f : D \to R$.
One to One:
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

One to One:

For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

or

Isomorphism principle.

Given a function, $f : D \to R$.
One to One:
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
or
$\forall x, y \in D$, $f(x) = f(y) \implies x = y$.

Isomorphism principle.

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Onto: For all $y \in R$, $\exists x \in D, y = f(x)$.

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$f(\cdot)$ is a **bijection** if it is one to one and onto.

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$f(\cdot)$ is a **bijection** if it is one to one and onto.

Isomorphism principle:
If there is a bijection $f : D \to R$ then $|D| = |R|$.

Countable.

How to count?

Countable.

How to count?

0,

Countable.

How to count?

0, 1,

Countable.

How to count?

0, 1, 2,

Countable.

How to count?

0, 1, 2, 3,

Countable.

How to count?

$0, 1, 2, 3, \ldots$

Countable.

How to count?

0, 1, 2, 3, $\ldots$

The Counting numbers.

Countable.

How to count?

0, 1, 2, 3, $\ldots$

The Counting numbers.
The natural numbers! N

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

Countable.

How to count?

0, 1, 2, 3, $\ldots$

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Where’s 0?

Which is bigger?

Where’s 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. 0,

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1,$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2,$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3,$

Where's 0?

Which is bigger?
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Natural numbers. $0, 1, 2, 3, \ldots$.

Where's 0?

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The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1,$

Where's 0?

Which is bigger?
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Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2,$

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3,$

Where's 0?

Which is bigger?
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Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

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But.. but Where's zero? "Comes from 1."

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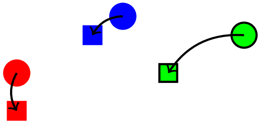

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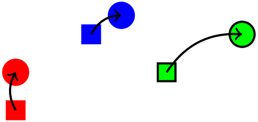

Inverse function!

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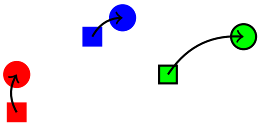

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Can prove equivalence either way.

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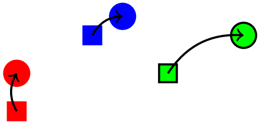

Inverse function!

Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

More large sets.

E - Even natural numbers?

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Evens are countably infinite.
Evens are same size as all natural numbers.

All integers?

What about Integers, Z?

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Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

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Integers and naturals have same size!

Listings..

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2	1
3	-2

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4	2

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If infinite: bijection with N.

Enumerability $\equiv$ countability.

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Enumerability $\equiv$ countability.

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61A

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61A —- streams!

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All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.

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$B = \{0,1\}^*$.

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Can't list in "order".

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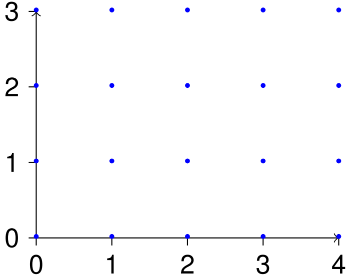

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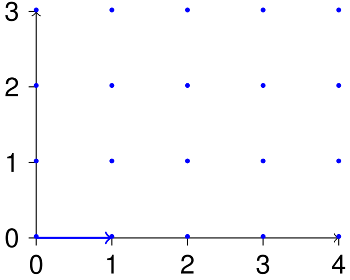

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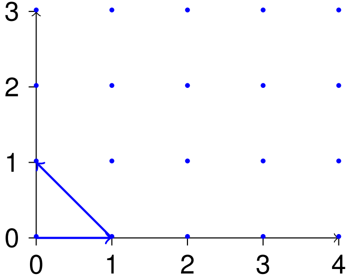

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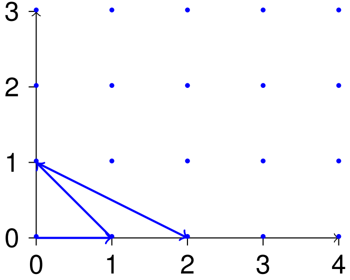

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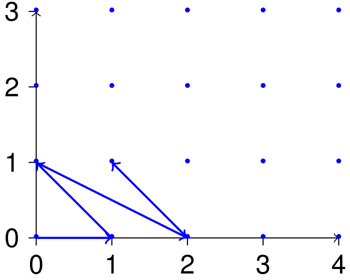

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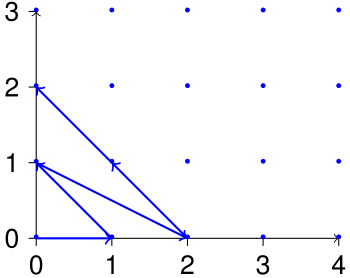

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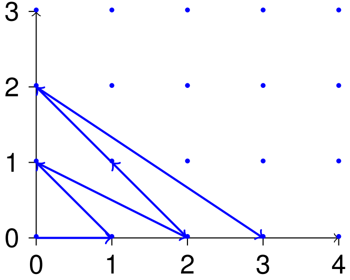

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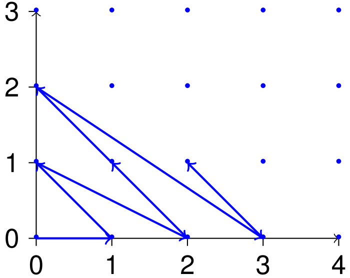

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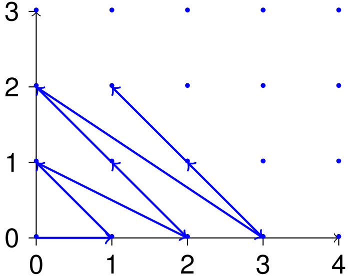

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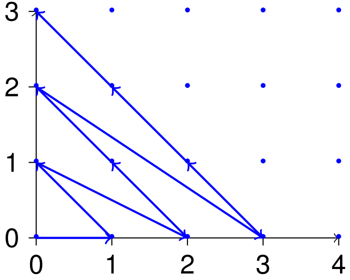

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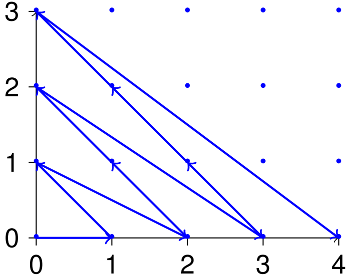

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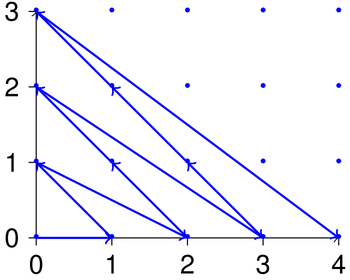

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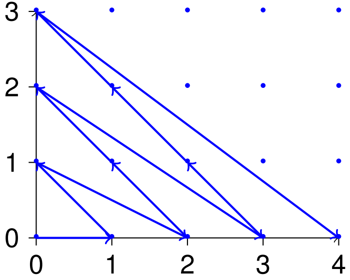

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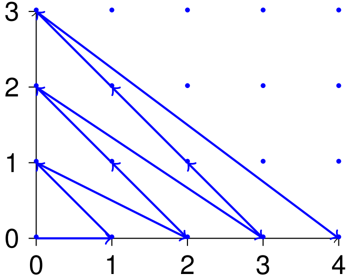

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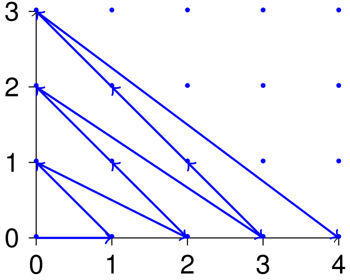

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Positive rational number.

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    Interleave Streams in 61A

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Interleave Streams in 61A

The rationals are countably infinite.

Real numbers..

Real numbers are same size as integers?

The reals.

Are the set of reals countable?

The reals.

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Lets consider the reals $[0, 1]$.

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Each real has a decimal representation.

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.345212312... Some real number

Diagonalization.

If countable, there a listing, L contains all reals.

Diagonalization.

If countable, there a listing, L contains all reals. For example

Diagonalization.

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0: .500000000...

Diagonalization.

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0: .500000000...
1: .785398162...

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...

Diagonalization.

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0: .500000000...
1: .785398162...
2: .367879441...
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Construct "diagonal" number:

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0: .500000000...
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2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .7

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .77

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct “diagonal” number: .776

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .7767

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .77677

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
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4: .345212312...
$\vdots$

Construct "diagonal" number: $.77677\ldots$

Diagonalization.

If countable, there a listing, L contains all reals. For example

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2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .77677 . . .

Diagonal Number:

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit i is 7 if number i's ith digit is not 7

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
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3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
and 6 otherwise.

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
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3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: $.77677\ldots$

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Diagonal number for a list differs from every number in list!

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Diagonal Number: Digit *i* is 7 if number *i*'s *i*th digit is not 7
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Diagonal number for a list differs from every number in list!
Diagonal number not in list.

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If countable, there a listing, L contains all reals. For example

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Diagonal Number: Digit i is 7 if number i's ith digit is not 7
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Diagonal number not in list.

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Contradiction!

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Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset $[0,1]$ is not countable!!

All reals?

Subset $[0, 1]$ is not countable!!

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.

If reals are countable then so is $[0, 1]$.

Diagonalization.

1. Assume that a set S can be enumerated.

Diagonalization.

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2. Consider an arbitrary list of all the elements of S.

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3. Use the diagonal from the list to construct a new element t.

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4. Show that t is different from all elements in the list

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   $\implies$ t is not in the list.

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5. Show that t is in S.
6. Contradiction.

Another diagonalization.

The set of all subsets of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,

Another diagonalization.

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Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds,

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Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds, primes,

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Assume is countable.

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Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
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Assume is countable.

There is a listing, L, that contains all subsets of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
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Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:

Another diagonalization.

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Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.

Another diagonalization.

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evens, odds, primes,

Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.
otherwise $i \notin D$.

D is different from ith set in L for every i.

Another diagonalization.

The set of all subsets of *N*.

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evens, odds, primes,

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If *i*th set in *L* does not contain *i*, $i \in D$.
otherwise $i \notin D$.

D is different from *i*th set in *L* for every *i*.
$\implies$ *D* is not in the listing.

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D is a subset of N.

Another diagonalization.

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Example subsets of *N*: $\{0\}$, $\{0,\ldots,7\}$,
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L does not contain all subsets of N.

Contradiction.

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Contradiction.

Theorem: The set of all subsets of N is not countable.

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evens, odds, primes,

Assume is countable.

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D is a subset of N.

L does not contain all subsets of N.

Contradiction.

Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S, is the **powerset** of N.)

Diagonalize Natural Number.

Natural numbers have a listing, L.

Diagonalize Natural Number.

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Make a diagonal number, D:
differ from ith element of L in ith digit.

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D is a natural number... Not.

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Any natural number has a finite number of digits.

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Differs from all elements of listing.

D is a natural number... Not.

Any natural number has a finite number of digits.

“Construction” requires an infinite number of digits.

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

First of Hilbert's problems!

Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

$f : R^+ \rightarrow [0, 1]$.

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \to [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

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One to one.

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One to one. $x \neq y$

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If both in $[0,1/2]$,

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If both in $[0,1/2]$, a shift

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Cardinality of $[0,1]$ smaller than all the reals?

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$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^{+} \rightarrow [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0,1/2]$

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Cardinality of $[0,1]$ smaller than all the reals?

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If one is in $[0,1/2]$ and one isn't,

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If both in $[0,1/2]$, a shift $\Longrightarrow f(x) \neq f(y)$.
If neither in $[0,1/2]$ a division $\Longrightarrow f(x) \neq f(y)$.
If one is in $[0,1/2]$ and one isn't, different ranges $\Longrightarrow f(x) \neq f(y)$.
Bijection!

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If one is in $[0,1/2]$ and one isn't, different ranges $\implies f(x) \neq f(y)$.
Bijection!

$[0,1]$ is same cardinality as nonnegative reals!

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.

Resolution of hypothesis?

Resolution of hypothesis?

Gödel. 1940.
Can’t use math!

Resolution of hypothesis?

Gödel. 1940.
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Uh oh....

Russell's Paradox.

Naive Set Theory: Any definable collection is a set.

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Axioms changed.

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See Logicomix by Doxiaidis, Papadimitriou (professor here),
Papadatos, Di Donna.

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Write me a program checker!

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Check that the compiler works!

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Run P on I and check!

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How long do you wait?

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Something about infinity here, maybe?

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(A) He is confused.

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(C). Maybe (D).

Halt and Turing.

Proof:

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2. Otherwise, halt immediately.

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Can run Turing on Turing!

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Assumption: there is a program HALT.
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Can run Turing on Turing!

Does Turing(Turing) halt?

Halt and Turing.

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Another view of proof: diagonalization.

Any program is a fixed length string.

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$$\begin{array}{c|cccc}
 & P_1 & P_2 & P_3 & \cdots \\
\hline
 & & & & \\
P_1 & \text{H} & \text{H} & \text{L} & \cdots \\
P_2 & \text{L} & \text{L} & \text{H} & \cdots \\
P_3 & \text{L} & \text{H} & \text{H} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
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\begin{array}{c|cccc}
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P_1 & \color{blue}{\text{H}} & \text{H} & \text{L} & \cdots \\
P_2 & \text{L} & \color{blue}{\text{L}} & \text{H} & \cdots \\
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P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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Proof play by play.

Assumed HALT(P, I) existed.

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What is P?

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Questions?

We are so smart!

Wow, that was easy!

We are so smart!

Wow, that was easy!

We should be famous!

No computers for Turing!

In Turing's time.

No computers for Turing!

In Turing's time.

No computers.

No computers for Turing!

In Turing's time.

No computers.

Adding machines.

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Adding machines.
e.g., Babbage (from table of logarithms) 1812.

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Adding machines.
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Concept of program as data wasn't really there.

Turing machine.

Turing machine.

A Turing machine.
– an (infinite) tape with characters

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– an (infinite) tape with characters
– be in a state, and read a character

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Universal Turing machine

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Now that's a computer!

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Turing: AI,

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Turing: AI, self modifying code,

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Now that's a computer!

Turing: AI, self modifying code, learning...

Turing and computing.

Just a mathematician?

Turing and computing.

Just a mathematician?

“Wrote” a chess program.

Turing and computing.

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“Wrote” a chess program.

Simulated the program by hand to play chess.

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It won!

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Involved with computing labs through the 40s.

Church, Gödel and Turing.

Church proved an equivalent theorem. (Previously.)

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Used λ calculus....

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Programming languages!

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Programming languages! javascript, ruby, python....

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Gödel: Incompleteness theorem.

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Any formal system either is inconsistent or incomplete.

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Along the way: "built" computers out of arithmetic.
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.... a natural number!

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Church proved an equivalent theorem. (Previously.)

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Computer, assembly code, programming language, browser, html, javascript..

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We can't get enough of building more Turing machines.

Undecidable problems.

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Undecidability for Diophantine set of equations
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- British Government apologized (2009) and pardoned (2013).

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Program is text, so we can pass it to itself,
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Summary: decidability.

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Computation is a lens for other action in the world.

Probability

What's to come?

Probability

What's to come? Probability.

Probability

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A bag contains:

Probability

What's to come? Probability.

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Probability

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A bag contains:

What is the chance that a ball taken from the bag is blue?

Probability

What's to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue.

Probability

What's to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total.

Probability

What's to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total. Divide.

Probability

What's to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total. Divide.

For now:

Probability

What's to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total. Divide.

For now: Counting!

Probability

What's to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total. Divide.

For now: Counting!

Later: Probability.