Today.
Today.

Farewell to modular arithmetic.
Farewell to modular arithmetic. Until the midterm.
Farewell to modular arithmetic. Until the midterm. And final.
Farewell to modular arithmetic. Until the midterm. And final. Coutability and Uncountability.
Today.

Farewell to modular arithmetic. Until the midterm. And final.
Coutability and Uncountability.
Undecidability.
Farewell (for now) to modular arithmetic...

Modular arithmetic modulo a prime.

Add, subtract, commutative, associative, inverses!

Allow for solving linear systems, discussing polynomials...

Why not modular arithmetic all the time? 4 \succ 3?

Yes!

4 \succ 3 (\text{mod} 7)?

Yes...

maybe?

\text{ Uh oh.. }

−3 \succ 3 (\text{mod} 7)?

Uh oh..

\text{−3} = 4 (\text{mod} 7).

Another problem.

4 is close to 3.

But can you get closer?

Sure.

3.5.

Closer.

Sure?

3.25, 3.1, 3.000001...

For real numbers we have the notion of limit, continuity, and derivative.......

....and Calculus.

For modular arithmetic...

no Calculus.

Sad face!
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$4 > 3$?
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$4 > 3$ ? Yes!

$-3 > 3$ (mod 7) ?

Yes...
maybe?

$-3$ = $4$ (mod 7).

Another problem.

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But can you get closer?

Sure.
$3.5$.
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Sure?
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Next up: how big is infinity.
Next up: how big is infinity.

- Countable
- Countably infinite.
- Enumeration
How big are the reals or the integers?

Infinite!
How big are the reals or the integers?

Infinite!

Is one bigger or smaller?
Same size?

- Circles $\rightarrow$ Squares.
  - $f(\text{red circle}) = \text{red square}$
  - $f(\text{blue circle}) = \text{blue square}$
  - $f(\text{circle with black border}) = \text{square with black border}$
  
  **One to One.** Each circle mapped to different square.
  
  **Onto.** Each square mapped to from some circle.
  
  **Isomorphism principle:** If there is $f: D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$. 
Same size?

Same number?

Make a function $f$: Circles $\rightarrow$ Squares.

- $f$(red circle) = red square
- $f$(blue circle) = blue square
- $f$(circle with black border) = square with black border

One to one. Each circle mapped to different square.

One to One: For all $x, y \in D$, $x \neq y \Rightarrow f(x) \neq f(y)$.

Onto. Each square mapped to from some circle.

Onto: For all $s \in R$, $\exists c \in D$, $s = f(c)$.

Isomorphism principle: If there is $f: D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$.
Same size?

Same number?
Make a function $f : \text{Circles} \rightarrow \text{Squares}$. 
Same size?

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One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
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Onto: For all \( s \in R \), \( \exists c \in D, s = f(c) \).

**Isomorphism principle:** If there is \( f : D \rightarrow R \) that is one to one and onto, then, \( |D| = |R| \).
Isomorphism principle.

Given a function, \( f : D \rightarrow R \).
Isomorphism principle.

Given a function, \( f : D \rightarrow R \).

**One to One:**
Isomorphism principle.

Given a function, $f : D \rightarrow R$.

**One to One:**
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

$f(\cdot)$ is a bijection if it is one to one and onto.

Isomorphism principle:
If there is a bijection $f : D \rightarrow R$ then $|D| = |R|$.
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$f(\cdot)$ is a **bijection** if it is one to one and onto.
Isomorphism principle.

Given a function, $f : D \to R$.

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For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

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If there is a bijection $f : D \rightarrow R$ then $|D| = |R|$. 
Countable.

How to count?
Countable.

How to count?

0,
Countable.

How to count?
0, 1,
How to count?
0, 1, 2,
Countable.

How to count?
0, 1, 2, 3,
Countable.

How to count?

0, 1, 2, 3, …
How to count?
0, 1, 2, 3, …
The Counting numbers.
Countable.

How to count?
0, 1, 2, 3, ...
The Counting numbers.
The natural numbers! $\mathbb{N}$
How to count?
0, 1, 2, 3, ...
The Counting numbers.
The natural numbers! $N$

Definition: $S$ is **countable** if there is a bijection between $S$ and some subset of $N$. 
Countable.

How to count?
0, 1, 2, 3, ...
The Counting numbers.
The natural numbers! \( N \)

Definition: \( S \) is **countable** if there is a bijection between \( S \) and some subset of \( N \).

If the subset of \( N \) is finite, \( S \) has finite **cardinality**.
How to count?
0, 1, 2, 3, …
The Counting numbers.
The natural numbers! $N$

Definition: $S$ is **countable** if there is a bijection between $S$ and some subset of $N$.
If the subset of $N$ is finite, $S$ has finite **cardinality**.
If the subset of $N$ is infinite, $S$ is **countably infinite**.
Where’s 0?

Which is bigger?
Where’s 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.
Where’s 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. 0,
Where’s 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. 0, 1,
Where’s 0?

Which is bigger?
The positive integers, \( \mathbb{Z}^+ \), or the natural numbers, \( \mathbb{N} \).

Natural numbers. 0, 1, 2,
Where's 0?

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The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. 0, 1, 2, 3,
Where’s 0?

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Natural numbers. 0, 1, 2, 3, ....
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Natural numbers. 0, 1, 2, 3, ....

Positive integers. 1,
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Natural numbers. 0, 1, 2, 3, ....

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Natural numbers. 0, 1, 2, 3, ....

Positive integers. 1, 2, 3,
Where’s 0?

Which is bigger?
The positive integers, \( \mathbb{Z}^+ \), or the natural numbers, \( \mathbb{N} \).

Natural numbers. 0, 1, 2, 3, …. 

Positive integers. 1, 2, 3, …. 

Consider \( f(z) = z - 1 \).

For any two \( z_1 \neq z_2 \), \( f(z_1) \neq f(z_2) \).

One to one!

For any natural number \( n \), for \( z = n + 1 \), \( f(z) = (n+1)-1 = n \).

Onto for \( \mathbb{N} \)

Bijection! \( \Rightarrow |\mathbb{Z}^+| = |\mathbb{N}| \).

But.. but Where’s zero? “Comes from 1.”
Where’s 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. 0, 1, 2, 3, ....

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Where’s 0?
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The positive integers, \( \mathbb{Z}^+ \), or the natural numbers, \( \mathbb{N} \).

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Positive integers. 1, 2, 3, ....

Where's 0?

More natural numbers!
Where’s 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. 0, 1, 2, 3, ....
Positive integers. 1, 2, 3, ....

Where’s 0?

More natural numbers!

Consider $f(z) = z - 1$. 
Where’s 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. 0, 1, 2, 3, ....

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More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2$
Where's 0?

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Natural numbers. 0, 1, 2, 3, ....
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Where's 0?
More natural numbers!
Consider $f(z) = z - 1$.
For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1$
Where’s 0?

Which is bigger?
The positive integers, \( \mathbb{Z}^+ \), or the natural numbers, \( \mathbb{N} \).

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For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.

One to one!
Where’s 0?

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For any natural number \( n \),
Where’s 0?

Which is bigger?
The positive integers, \( \mathbb{Z}^+ \), or the natural numbers, \( \mathbb{N} \).

Natural numbers. 0, 1, 2, 3, ....

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Consider \( f(z) = z - 1 \).

For any two \( z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2) \).

One to one!

For any natural number \( n \), for \( z = n + 1 \),
Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. 0, 1, 2, 3, ….

Positive integers. 1, 2, 3, ….

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$. One to one!

For any natural number $n$, for $z = n + 1$, $f(z)$
Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. 0, 1, 2, 3, ....

Positive integers. 1, 2, 3, ....

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.

One to one!

For any natural number $n$, for $z = n + 1$, $f(z) = (n + 1) - 1$
Where’s 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. 0, 1, 2, 3, ....

Positive integers. 1, 2, 3, ....

Where’s 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.

One to one!

For any natural number $n$, for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.  

Where’s 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. 0, 1, 2, 3, ....

Positive integers. 1, 2, 3, ....

Where’s 0?

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One to one!

For any natural number $n$, for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.

Onto for $\mathbb{N}$
Where’s 0?

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Consider $f(z) = z - 1$.

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One to one!

For any natural number $n$, for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.

Onto for $\mathbb{N}$

Bijection!
Where’s 0?

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Natural numbers. 0, 1, 2, 3, ....

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More natural numbers!

Consider \( f(z) = z - 1 \).

For any two \( z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2) \).

One to one!

For any natural number \( n \), for \( z = n + 1 \), \( f(z) = (n + 1) - 1 = n \).

Onto for \( \mathbb{N} \)

Bijection! \( \implies |\mathbb{Z}^+| = |\mathbb{N}|. \)
Where’s 0?

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Positive integers. 1, 2, 3, ….  
Where’s 0?  
More natural numbers!  
Consider \( f(z) = z - 1 \).

For any two \( z_1 \neq z_2 \) \( \implies z_1 - 1 \neq z_2 - 1 \) \( \implies f(z_1) \neq f(z_2) \).
One to one!  
For any natural number \( n \), for \( z = n + 1 \), \( f(z) = (n + 1) - 1 = n \).
Onto for \( \mathbb{N} \)  
Bijection! \( \implies |\mathbb{Z}^+| = |\mathbb{N}|. \)
But.. but
Where's 0?

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Bijection! $\implies |\mathbb{Z}^+| = |\mathbb{N}|$.

But.. but Where’s zero?
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For any natural number $n$, for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.

Onto for $\mathbb{N}$

Bijection! $\implies |\mathbb{Z}^+| = |\mathbb{N}|$.

But.. but Where’s zero? “Comes from 1.”
A bijection is a bijection.
A bijection is a bijection.

Notice that there is a bijection between \( N \) and \( Z^+ \) as well.
A bijection is a bijection.

Notice that there is a bijection between $N$ and $Z^+$ as well. $f(n) = n + 1$. 
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$$f(n) = n + 1. \ 0 \to 1,$$
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Notice that there is a bijection between $N$ and $Z^+$ as well. $f(n) = n + 1$. $0 \rightarrow 1, 1 \rightarrow 2,$
A bijection is a bijection.

Notice that there is a bijection between $N$ and $Z^+$ as well. $f(n) = n + 1$. 0 $\rightarrow$ 1, 1 $\rightarrow$ 2, …
A bijection is a bijection.

Notice that there is a bijection between $N$ and $\mathbb{Z}^+$ as well.

$f(n) = n + 1. \ 0 \rightarrow 1, 1 \rightarrow 2, \ldots$

Bijection from $A$ to $B \iff$ a bijection from $B$ to $A$. 
A bijection is a bijection.

Notice that there is a bijection between $N$ and $\mathbb{Z}^+$ as well.

$$f(n) = n + 1. \ 0 \rightarrow 1, \ 1 \rightarrow 2, \ldots$$

Bijection from $A$ to $B \implies$ a bijection from $B$ to $A.$
A bijection is a bijection.

Notice that there is a bijection between $N$ and $Z^+$ as well.
$f(n) = n + 1. \ 0 \rightarrow 1, \ 1 \rightarrow 2, \ldots$

Bijection from $A$ to $B \implies$ a bijection from $B$ to $A$.

Inverse function!
A bijection is a bijection.

Notice that there is a bijection between \( N \) and \( \mathbb{Z}^+ \) as well.
\[ f(n) = n + 1. \quad 0 \rightarrow 1, \, 1 \rightarrow 2, \, \ldots \]

Bijection from \( A \) to \( B \) \( \implies \) a bijection from \( B \) to \( A \).

Inverse function!

Can prove equivalence either way.
A bijection is a bijection.

Notice that there is a bijection between \( N \) and \( \mathbb{Z}^+ \) as well. 
\[
 f(n) = n + 1. \ 0 \rightarrow 1, \ 1 \rightarrow 2, \ldots
\]

Bijection from \( A \) to \( B \) \( \implies \) a bijection from \( B \) to \( A \).

Inverse function!

Can prove equivalence either way. 
Bijection to or from natural numbers implies countably infinite.
More large sets.

$E$ - Even natural numbers?
More large sets.

$E$ - Even natural numbers?

$f : \mathbb{N} \rightarrow E.$
More large sets.

\[ E - \text{Even natural numbers?} \]
\[ f : \mathbb{N} \to E. \]
\[ f(n) \to 2n. \]
More large sets.

$E$ - Even natural numbers?

$f : \mathbb{N} \rightarrow E.$

$f(n) \rightarrow 2n.$

Onto:
More large sets.

$E$ - Even natural numbers?

$f : \mathbb{N} \rightarrow E.$

$f(n) \rightarrow 2n.$

Onto: $\forall e \in E,$ $f(e/2) = e.$
More large sets.

$E$ - Even natural numbers?

$f : \mathbb{N} \rightarrow E$.

$f(n) \rightarrow 2n$.

Onto: $\forall e \in E$, $f(e/2) = e$. $e/2$ is natural since $e$ is even.
$E$ - Even natural numbers?

$f : \mathbb{N} \rightarrow E$.

$f(n) \rightarrow 2n$.

Onto: $\forall e \in E, f(e/2) = e$. $e/2$ is natural since $e$ is even

One-to-one:
More large sets.

$E$ - Even natural numbers?

$f : \mathbb{N} \rightarrow E.$

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One-to-one: $\forall x, y \in \mathbb{N}, x \neq y \implies 2x \neq 2y.$
More large sets.

\( E \) - Even natural numbers?

\( f : \mathbb{N} \to E. \)

\( f(n) \to 2n. \)

Onto: \( \forall e \in E, f(e/2) = e. \) \( e/2 \) is natural since \( e \) is even

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Evens are countably infinite.
More large sets.

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Evens are countably infinite.

Evens are same size as all natural numbers.
All integers?

What about Integers, $\mathbb{Z}$?
All integers?

What about Integers, \( Z \)?
Define \( f : N \to Z \).

\[
 f(n) = \begin{cases} 
 n/2 & \text{if } n \text{ even} \\
 -(n+1)/2 & \text{if } n \text{ odd.}
\end{cases}
\]
All integers?

What about Integers, $\mathbb{Z}$?
Define $f : \mathbb{N} \rightarrow \mathbb{Z}$.

$$f(n) = \begin{cases} 
  \frac{n}{2} & \text{if } n \text{ even} \\
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One-to-one: For $x \neq y$
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One-to-one: For \( x \neq y \)
if \( x \) is even and \( y \) is odd,
All integers?

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Define $f : \mathbb{N} \to \mathbb{Z}$.

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One-to-one: For $x \neq y$
if $x$ is even and $y$ is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative
All integers?

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All integers?

What about Integers, $\mathbb{Z}$?
Define $f : \mathbb{N} \rightarrow \mathbb{Z}$.

$$f(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ even} \\
-(\frac{n+1}{2}) & \text{if } n \text{ odd.}
\end{cases}$$

One-to-one: For $x \neq y$
if $x$ is even and $y$ is odd, then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if $x$ is even and $y$ is even, then $x/2 \neq y/2 \implies f(x) \neq f(y)$

Integers and naturals have same size!
All integers?

What about Integers, \( \mathbb{Z} \)?
Define \( f : \mathbb{N} \to \mathbb{Z} \).

\[
f(n) = \begin{cases} 
  n/2 & \text{if } n \text{ even} \\ 
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\[\ldots\]
All integers?

What about Integers, $\mathbb{Z}$? Define $f : \mathbb{N} \rightarrow \mathbb{Z}$.

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....
All integers?

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One-to-one: For \( x \neq y \)
if \( x \) is even and \( y \) is odd,
then \( f(x) \) is nonnegative and \( f(y) \) is negative \( \Rightarrow f(x) \neq f(y) \)
if \( x \) is even and \( y \) is even,
then \( x/2 \neq y/2 \) \( \Rightarrow f(x) \neq f(y) \)

\[
\ldots
\]

Onto: For any \( z \in Z \),
All integers?

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Onto: For any \( z \in Z \),
if \( z \geq 0 \), \( f(2z) = z \) and \( 2z \in N \).
All integers?

What about Integers, \( \mathbb{Z} \)?
Define \( f : \mathbb{N} \rightarrow \mathbb{Z} \).

\[
\begin{align*}
    f(n) &= \begin{cases} 
        n/2 & \text{if } n \text{ even} \\
        -(n+1)/2 & \text{if } n \text{ odd.}
    \end{cases}
\end{align*}
\]

One-to-one: For \( x \neq y \)
if \( x \) is even and \( y \) is odd,
then \( f(x) \) is nonnegative and \( f(y) \) is negative \( \implies f(x) \neq f(y) \)
if \( x \) is even and \( y \) is even,
then \( x/2 \neq y/2 \implies f(x) \neq f(y) \)
....

Onto: For any \( z \in \mathbb{Z} \),
if \( z \geq 0 \), \( f(2z) = z \) and \( 2z \in \mathbb{N} \).
if \( z < 0 \), \( f(2|z|-1) = z \) and \( 2|z|+1 \in \mathbb{N} \).
All integers?

What about Integers, $\mathbb{Z}$?
Define $f : \mathbb{N} \rightarrow \mathbb{Z}$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if } n \text{ odd}. \end{cases}$$

One-to-one: For $x \neq y$
if $x$ is even and $y$ is odd,
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if $x$ is even and $y$ is even,
then $x/2 \neq y/2 \implies f(x) \neq f(y)$
....

Onto: For any $z \in \mathbb{Z}$,
if $z \geq 0$, $f(2z) = z$ and $2z \in \mathbb{N}$.
if $z < 0$, $f(2|z| - 1) = z$ and $2|z| + 1 \in \mathbb{N}$.

Integers and naturals have same size!
Listings..

\[ f(n) = \begin{cases} 
  n/2 & \text{if } n \text{ even} \\
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Listings..

\[ f(n) = \begin{cases} 
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Another View:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f(n) )</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>( -1 )</td>
</tr>
<tr>
<td>2</td>
<td>( 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( -2 )</td>
</tr>
<tr>
<td>4</td>
<td>( 2 )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Notice that: A listing “is” a bijection with a subset of natural numbers. Function \( f \equiv \) “Position in list.” If finite: bijection with \( \{0, \ldots, |S| - 1\} \). If infinite: bijection with \( \mathbb{N} \).
Listings..

\[ f(n) = \begin{cases} 
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$$f(n) = \begin{cases} 
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Listings..

\[ f(n) = \begin{cases} 
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-(\frac{n+1}{2}) & \text{if } n \text{ odd.}
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Notice that: A listing “is” a bijection with a subset of natural numbers.

Function \( \equiv \) “Position in list.”

If finite: bijection with \( \{0, \ldots, |S| - 1\} \)

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If finite: bijection with \(\{0, \ldots, |S| - 1\}\)
If infinite: bijection with \(\mathbb{N}\).
Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.
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“Output element of $S$”,

$Z = \{0, 1, -1, 2, -2, \ldots\}$

When do you get to $-1$ at infinity?

Need to be careful.

61A— streams!
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When do you get to −1? at infinity?

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61A —- streams!
Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset $T$ of a countable set $S$ is countable.
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Enumerate $T$ as follows:  
Get next element, $x$, of $S$,  

$\mathbb{Z}^+$ is countable.  
It is infinite since the list goes on.  
There is a bijection with the natural numbers.  
So it is countably infinite.  
All countably infinite sets have the same cardinality.
Countably infinite subsets.

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Corollary: Any subset $T$ of a countable set $S$ is countable. 

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All countably infinite sets have the same cardinality.
All binary strings.
Enumeration example.

All binary strings.
\[ B = \{0, 1\}^*. \]
All binary strings.
$B = \{0, 1\}^*$. 
$B = \{\phi, \}$
Enumeration example.

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$B = \{0, 1\}^*.$

$B = \{\phi, 0, 1\}^*.$

Never get to 1.
Enumeration example.

All binary strings.  
\[ B = \{0, 1\}^*. \]
\[ B = \{\phi, 0, 1, \ldots\}. \]
\[ \phi \] is empty string.  
For any string, it appears at some position in the list.  
If \( n \) bits, it will appear before position \( 2^n + 1 \).  
Should be careful here.
Enumeration example.

All binary strings.

\( B = \{0, 1\}^* \).

\( B = \{\emptyset, 0, 1, 00, \ldots\} \).

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All binary strings.

\[ B = \{0, 1\}^*. \]

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All binary strings.
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\( B = \{\phi; 0, 00, 000, 0000, \ldots\} \)

Never get to 1.
More fractions?

Enumerate the rational numbers in order...
More fractions?

Enumerate the rational numbers in order...
0,...,1/2,..
More fractions?

Enumerate the rational numbers in order...
0, ..., 1/2, ..
Where is 1/2 in list?
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Where is 1/2 in list?
After 1/3, which is after 1/4, which is after 1/5...
More fractions?

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A thing about fractions:
More fractions?

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A thing about fractions:
any two fractions has another fraction between it.
More fractions?

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Can’t even get to “next” fraction!
Enumerate the rational numbers in order...
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Where is 1/2 in list?

After 1/3, which is after 1/4, which is after 1/5...

A thing about fractions:
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Can’t even get to “next” fraction!

Can’t list in “order”.
Pairs of natural numbers.

Consider pairs of natural numbers: \( N \times N \)
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E.g.: $(1, 2)$, $(100, 30)$, etc.
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For finite sets $S_1$ and $S_2$, then $S_1 \times S_2$
Consider pairs of natural numbers: $N \times N$

E.g.: $(1, 2), (100, 30)$, etc.

For finite sets $S_1$ and $S_2$, then $S_1 \times S_2$ has size $|S_1| \times |S_2|$.
Consider pairs of natural numbers: $N \times N$
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Pairs of natural numbers.

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So, $N \times N$ is countably infinite
Consider pairs of natural numbers: $N \times N$
E.g.: $(1, 2), (100, 30)$, etc.

For finite sets $S_1$ and $S_2$, then $S_1 \times S_2$ has size $|S_1| \times |S_2|$.

So, $N \times N$ is countably infinite squared.
Pairs of natural numbers.

Consider pairs of natural numbers: \( N \times N \)
E.g.: \((1, 2), (100, 30), \) etc.

For finite sets \( S_1 \) and \( S_2 \),
then \( S_1 \times S_2 \)
has size \( |S_1| \times |S_2| \).

So, \( N \times N \) is countably infinite squared ??
Pairs of natural numbers.

Enumerate in list:

- (0, 0)
- (1, 0)
- (0, 1)
- (2, 0)
- (1, 1)
- (0, 2)

...(more pairs)...

The pair \((a, b)\), is in first \(\approx (a + b + 1)(a + b)/2\) elements of list! (i.e., "triangle").

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:
(0,0),

The pair (a, b), is in list $(a+b+1)(a+b)/2$ elements of list!

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Pairs of natural numbers.

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The pair (a, b), is in first \( \frac{a+b+1}{2} \) elements of list!

Countably infinite.
Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:

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Pairs of natural numbers.

Enumerate in list:

\((0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \ldots\)

The pair \((a,b)\), is in first 
\((a+b+1)(a+b)/2\) elements of list!

(i.e., “triangle”).

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:

\[(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \ldots.\]
Pairs of natural numbers.

Enumerate in list:

\[(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \ldots\]

The pair \((a, b)\), is in first \(\approx \frac{a + b + 1}{2}\) elements of list!

(i.e., "triangle").

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:
(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), ……

The pair \((a, b)\), is in first \(\approx (a + b + 1)(a + b)/2\) elements of list!

(i.e., “triangle”).

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:
\((0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \ldots\)

\[\text{The pair } (a, b), \text{ is in first } \approx (a + b + 1)(a + b)/2 \text{ elements of list!}\]
Pairs of natural numbers.

Enumerate in list:
(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), …

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Pairs of natural numbers.

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Countably infinite.
Pairs of natural numbers.

Enumerate in list:
(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), …

The pair \((a, b)\), is in first \(\approx (a + b + 1)(a + b)/2\) elements of list!
(i.e., “triangle”).

Countably infinite.

Same size as the natural numbers!!
Rationals?

Positive rational number.
Positive rational number. 
Lowest terms: $a/b$
Rationals?

Positive rational number.
Lowest terms: \( a/b \)
\( a, b \in N \)
Rationals?

Positive rational number.
Lowest terms: \( a/b \)
\( a, b \in \mathbb{N} \)
with \( \gcd(a, b) = 1 \).
Rationals?

Positive rational number.
Lowest terms: \( a/b \)
\( a, b \in N \)
with \( \gcd(a, b) = 1 \).

Infinite subset of \( N \times N \).
Rationals?

Positive rational number.
Lowest terms: \( a/b \)
\( a, b \in N \)
with \( \gcd(a, b) = 1 \).

Infinite subset of \( N \times N \).

Countably infinite!
Rationals?

Positive rational number.
Lowest terms: $a/b$
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?
Rationals?

Positive rational number.
Lowest terms: \( a/b \)
\[ a, b \in \mathbb{N} \]
with \( \gcd(a, b) = 1 \).

Infinite subset of \( \mathbb{N} \times \mathbb{N} \).

Countably infinite!

All rational numbers?

Negative rationals are countable.
Rationals?

Positive rational number.
Lowest terms: \( a/b \)
\( a, b \in N \)
with \( \gcd(a, b) = 1 \).

Infinite subset of \( N \times N \).

Countably infinite!

All rational numbers?
Negative rationals are countable. (Same size as positive rationals.)
Rationals?

Positive rational number.
Lowest terms: \( \frac{a}{b} \)
\( a, b \in N \)
with \( \gcd(a, b) = 1 \).

Infinite subset of \( N \times N \).

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.
Rationals?

Positive rational number.
Lowest terms: $a/b$
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative
Rationals?

Positive rational number.  
Lowest terms: $a/b$  
$a, b \in N$  
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative ???
Rationals?

Positive rational number. Lowest terms: \( \frac{a}{b} \), \( a, b \in \mathbb{N} \) with \( \gcd(a, b) = 1 \).

Infinite subset of \( \mathbb{N} \times \mathbb{N} \).

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative ??? No!
Rationals?

Positive rational number.
Lowest terms: $a/b$
$a,b \in N$
with $gcd(a,b) = 1$.

Infinite subset of $N \times N$.
Countably infinite!

All rational numbers?
Negative rationals are countable. (Same size as positive rationals.)
Put all rational numbers in a list.
First negative, then nonegative ??? No!
Repeatedly and alternatively take one from each list.
Rationals?

Positive rational number.
Lowest terms: $a/b$
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.
Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)
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First negative, then nonegative ??? No!

Repeatedly and alternatively take one from each list.
Interleave Streams in 61A
Positive rational number.

Lowest terms: $a/b$
$a, b \in N$
with $gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative ??? No!

Repeatedly and alternatively take one from each list.

Interleave Streams in 61A

The rationals are countably infinite.
Real numbers.

Real numbers are same size as integers?
The reals.

Are the set of reals countable?
The reals.

Are the set of reals countable?
Lets consider the reals [0, 1].
The reals.

Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.
The reals.

Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

$0.5000000000...$
The reals.

Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

$.500000000\ldots$ (1/2)
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

$.500000000...$ (1/2)

$.785398162...$
Are the set of reals countable?

Let's consider the reals \([0, 1]\).

Each real has a decimal representation.

- \(0.500000000\ldots\) (\(1/2\))
- \(0.785398162\ldots\) (\(\pi/4\))
The reals.

Are the set of reals countable?

Let's consider the reals [0, 1].

Each real has a decimal representation.

.500000000... (1/2)
.785398162... π/4
.367879441...
Are the set of reals countable?

Let's consider the reals [0, 1].

Each real has a decimal representation.

.500000000... (1/2)
.785398162... π/4
.367879441... 1/e
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

- $0.500000000...$ (1/2)
- $0.785398162...$ $\pi/4$
- $0.367879441...$ $1/e$
- $0.632120558...$
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

- $0.500000000...$ $(1/2)$
- $0.785398162...$ $\pi/4$
- $0.367879441...$ $1/e$
- $0.632120558...$ $1 - 1/e$
The reals.

Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

- $.500000000...$ (1/2)
- $.785398162...$ $\pi/4$
- $.367879441...$ $1/e$
- $.632120558...$ $1 - 1/e$
- $.345212312...$
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

$.500000000\ldots$ (1/2)
$.785398162\ldots$ $\pi/4$
$.367879441\ldots$ $1/e$
$.632120558\ldots$ $1 – 1/e$
$.345212312\ldots$ Some real number
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

- $0.500000000...$ (1/2)
- $0.785398162...$ $\pi/4$
- $0.367879441...$ $1/e$
- $0.632120558...$ $1 - 1/e$
- $0.345212312...$ Some real number
Diagonalization.

If countable, there a listing, $L$ contains all reals.
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example
0: .5000000000...

Construct "diagonal" number:

Diagonal Number:

Digit $i$ is 7 if number $i$'s $i$th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!

Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset $[0, 1]$ is not countable!!
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example
0: .500000000...
1: .785398162...
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \(.500000000\ldots\)
1: \(.785398162\ldots\)
2: \(.367879441\ldots\)
3: \(.632120558\ldots\)
4: \(.345212312\ldots\)

Construct “diagonal” number:

\[
\begin{array}{cccccccc}
7 & 7 & 6 & 7 & \ldots
\end{array}
\]

Diagonal Number:

Digit \( i \) is 7 if number \( i \)'s \( i \)th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.
Diagonal number is real.
Contradiction!
Subset \([0, 1]\) is not countable!!
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

...
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number:
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .7
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

:\

Construct “diagonal” number: .776
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: \textcolor{red}{.5}000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .7767
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

: 

Construct “diagonal” number: .77677
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77677…
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77677…

Diagonal Number:
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \(.500000000\ldots\)
1: \(.785398162\ldots\)
2: \(.367879441\ldots\)
3: \(.632120558\ldots\)
4: \(.345212312\ldots\)

Construct “diagonal” number: \(.77677\ldots\)

Diagonal Number: Digit \( i \) is 7 if number \( i \)’s \( i \)th digit is not 7
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
...

Construct “diagonal” number: .77677...

Diagonal Number:Digit $i$ is 7 if number $i$’s $i$th digit is not 7 and 6 otherwise.
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: $.5000000000...$

1: $.785398162...$

2: $.367879441...$

3: $.632120558...$

4: $.345212312...$

...

Construct “diagonal” number: $.77677...$

Diagonal Number: Digit $i$ is 7 if number $i$’s $i$th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77677...

Diagonal Number: Digit $i$ is 7 if number $i$’s $i$th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list! Diagonal number not in list.
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77677…

Diagonal Number: Digit $i$ is 7 if number $i$’s $i$th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list! Diagonal number not in list.

Diagonal number is real.
Diagonalization.

If countable, there a listing, *L contains all reals*. For example

0: \(.500000000\ldots\)
1: \(.785398162\ldots\)
2: \(.367879441\ldots\)
3: \(.632120558\ldots\)
4: \(.345212312\ldots\)

Construct “diagonal” number: \(.77677\ldots\)

**Diagonal Number**: Digit *i* is 7 if number *i*’s *i*th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!

**Diagonal number not in list.**

**Diagonal number is real.**

**Contradiction!**
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \(.500000000...\)
1: \(.785398162...\)
2: \(.367879441...\)
3: \(.632120558...\)
4: \(.345212312...\)

Construct “diagonal” number: \(.77677...\)

Diagonal Number: Digit \( i \) is 7 if number \( i \)’s \( i \)th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!

Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset \([0,1]\) is not countable!!
All reals?

Subset [0, 1] is not countable!!
All reals?

Subset \([0, 1]\) is not countable!!

What about all reals?
All reals?

Subset $[0,1]$ is not countable!!

What about all reals?
No.
All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.
All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.
If reals are countable then so is $[0, 1]$. 
Diagonalization.

1. Assume that a set $S$ can be enumerated.
Diagonalization.

1. Assume that a set $S$ can be enumerated.
2. Consider an arbitrary list of all the elements of $S$. 
Diagonalization.

1. Assume that a set $S$ can be enumerated.
2. Consider an arbitrary list of all the elements of $S$.
3. Use the diagonal from the list to construct a new element $t$. 
4. Show that $t$ is different from all elements in the list $= \implies t$ is not in the list.
5. Show that $t$ is in $S$.
6. Contradiction.
1. Assume that a set $S$ can be enumerated.
2. Consider an arbitrary list of all the elements of $S$.
3. Use the diagonal from the list to construct a new element $t$.
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5. Show that $t$ is in $S$.
6. Contradiction.
Diagonalization.

1. Assume that a set $S$ can be enumerated.
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4. Show that $t$ is different from all elements in the list $\implies t$ is not in the list.
1. Assume that a set $S$ can be enumerated.
2. Consider an arbitrary list of all the elements of $S$.
3. Use the diagonal from the list to construct a new element $t$.
4. Show that $t$ is different from all elements in the list $\Rightarrow t$ is not in the list.
5. Show that $t$ is in $S$. 
6. Contradiction.
Diagonalization.

1. Assume that a set \( S \) can be enumerated.
2. Consider an arbitrary list of all the elements of \( S \).
3. Use the diagonal from the list to construct a new element \( t \).
4. Show that \( t \) is different from all elements in the list \( \implies t \) is not in the list.
5. Show that \( t \) is in \( S \).
6. Contradiction.
Another diagonalization.

The set of all subsets of $N$. 
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\}, \)

\( \{0, \ldots, 7\} \), evens, odds, primes,
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0, \ldots, 7\}$,
Another diagonalization.

The set of all subsets of $\mathbb{N}$.

Example subsets of $\mathbb{N}$: $\{0\}$, $\{0,\ldots,7\}$,
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\} \), \( \{0, \ldots, 7\} \), evens,
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds,
Another diagonalization.

The set of all subsets of \( N \).

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Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: \{0\}, \{0, \ldots, 7\},

evens, odds, primes,
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0,\ldots,7\}$, evens, odds, primes,

Assume is countable.
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: \{0\}, \{0,\ldots,7\}, evens, odds, primes,

Assume is countable.

There is a listing, $L$, that contains all subsets of $N$. 
Another diagonalization.

The set of all subsets of $\mathbb{N}$.

Example subsets of $\mathbb{N}$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes,

Assume is countable.

There is a listing, $L$, that contains all subsets of $\mathbb{N}$.

Define a diagonal set, $D$:
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\} \), \( \{0, \ldots, 7\} \), evens, odds, primes,

Assume is countable.

There is a listing, \( L \), that contains all subsets of \( N \).

Define a diagonal set, \( D \):
If \( i \)th set in \( L \) does not contain \( i \), \( i \in D \).
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes,

Assume is countable.

There is a listing, $L$, that contains all subsets of $N$.

Define a diagonal set, $D$:
If $i$th set in $L$ does not contain $i$, $i \in D$.
otherwise $i \notin D$.

$L$ does not contain all subsets of $N$.
Contradiction.

Theorem: The set of all subsets of $N$ is not countable.
(The set of all subsets of $S$, is the powerset of $N$.)
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0,\ldots,7\}$, evens, odds, primes,

Assume is countable.

There is a listing, $L$, that contains all subsets of $N$.

Define a diagonal set, $D$:
If $i$th set in $L$ does not contain $i$, $i \in D$.
otherwise $i \notin D$.

$D$ is different from $i$th set in $L$ for every $i$.
$= \Rightarrow D$ is not in the listing.

$L$ does not contain all subsets of $N$.
Contradiction.

Theorem:
The set of all subsets of $N$ is not countable.
(The set of all subsets of $S$, is the powerset of $N$.)

Another diagonalization.

The set of all subsets of $\mathbb{N}$.

Example subsets of $\mathbb{N}$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes,

Assume is countable.

There is a listing, $L$, that contains all subsets of $\mathbb{N}$.

Define a diagonal set, $D$:
If $i$th set in $L$ does not contain $i$, $i \in D$.
otherwise $i \notin D$.

$D$ is different from $i$th set in $L$ for every $i$. 

(The set of all subsets of $\mathbb{S}$, is the powerset of $\mathbb{N}$.)
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\} \), \( \{0, \ldots, 7\} \),
evens, odds, primes,

Assume is countable.

There is a listing, \( L \), that contains all subsets of \( N \).

Define a diagonal set, \( D \):
If \( i \)th set in \( L \) does not contain \( i \), \( i \in D \).
otherwise \( i \not\in D \).

\( D \) is different from \( i \)th set in \( L \) for every \( i \).
\( \implies \) \( D \) is not in the listing.
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, $L$, that contains all subsets of $N$.

Define a diagonal set, $D$:
If $i$th set in $L$ does not contain $i$, $i \in D$.
otherwise $i \notin D$.

$D$ is different from $i$th set in $L$ for every $i$.
$\implies D$ is not in the listing.

$D$ is a subset of $N$. 
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\} \), \( \{0,\ldots,7\} \), evens, odds, primes,

Assume is countable.

There is a listing, \( L \), that contains all subsets of \( N \).

Define a diagonal set, \( D \):
If \( i \)th set in \( L \) does not contain \( i \), \( i \in D \).
otherwise \( i \notin D \).

\( D \) is different from \( i \)th set in \( L \) for every \( i \).
\( \implies \) \( D \) is not in the listing.

\( D \) is a subset of \( N \).

\( L \) does not contain all subsets of \( N \).
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes,

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There is a listing, $L$, that contains all subsets of $N$.

Define a diagonal set, $D$:
If $i$th set in $L$ does not contain $i$, $i \in D$.
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$D$ is different from $i$th set in $L$ for every $i$.
$\implies D$ is not in the listing.

$D$ is a subset of $N$.
$L$ does not contain all subsets of $N$.
Contradiction.
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\} \), \( \{0,\ldots,7\} \), evens, odds, primes,

Assume is countable.

There is a listing, \( L \), that contains all subsets of \( N \).

Define a diagonal set, \( D \):
If \( i \)th set in \( L \) does not contain \( i \), \( i \in D \).
otherwise \( i \notin D \).

\( D \) is different from \( i \)th set in \( L \) for every \( i \).
\[ \implies D \] is not in the listing.

\( D \) is a subset of \( N \).

\( L \) does not contain all subsets of \( N \).

Contradiction.

**Theorem:** The set of all subsets of \( N \) is not countable.
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\} \), \( \{0,\ldots,7\} \),
evens, odds, primes,

Assume is countable.

There is a listing, \( L \), that contains all subsets of \( N \).

Define a diagonal set, \( D \):
If \( i \)th set in \( L \) does not contain \( i \), \( i \in D \).
otherwise \( i \notin D \).

\( D \) is different from \( i \)th set in \( L \) for every \( i \).
\( \implies D \) is not in the listing.

\( D \) is a subset of \( N \).

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Contradiction.

**Theorem:** The set of all subsets of \( N \) is not countable.
(The set of all subsets of \( S \), is the powerset of \( N \).)
Diagonalize Natural Number.

Natural numbers have a listing, \( L \).
Diagonalize Natural Number.

Natural numbers have a listing, $L$.

Make a diagonal number, $D$:

differ from $i$th element of $L$ in $i$th digit.
Natural numbers have a listing, \( L \).
Make a diagonal number, \( D \):
differ from \( i \)th element of \( L \) in \( i \)th digit.
Differs from all elements of listing.
Diagonalize Natural Number.

Natural numbers have a listing, \( L \).

Make a diagonal number, \( D \):

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Differs from all elements of listing.

\( D \) is a natural number...
Diagonalize Natural Number.

Natural numbers have a listing, $L$.

Make a diagonal number, $D$:

- differ from $i$th element of $L$ in $i$th digit.

- Differs from all elements of listing.

$D$ is a natural number... Not.
Diagonalize Natural Number.

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Any natural number has a finite number of digits.
Natural numbers have a listing, $L$.

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Differs from all elements of listing.

$D$ is a natural number... Not.

Any natural number has a finite number of digits.

“Construction” requires an infinite number of digits.
The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.
The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals. First of Hilbert’s problems!
Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?
Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

$f : \mathbb{R}^+ \rightarrow [0, 1]$. 
Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

$f : R^+ \rightarrow [0, 1]$. 

$$f(x) = \begin{cases} 
    x + \frac{1}{2} & \text{for } 0 \leq x \leq 1/2 \\
    \frac{1}{4x} & \text{for } x > 1/2 
\end{cases}$$
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One to one.
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If one is in $[0, 1/2]$ and one isn’t, different ranges
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Bijection!
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If one is in $[0, 1/2]$ and one isn’t, different ranges $\implies f(x) \neq f(y)$.

Bijection!

$[0, 1]$ is same cardinality as nonnegative reals!
Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.
Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.
Resolution of hypothesis?

Gödel. 1940.

Can't use math!

If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Who shaves the barber?

Self reference.

Can a program refer to a program?

Can a program refer to itself?

Uh oh....
Resolution of hypothesis?

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Uh oh....
Russell’s Paradox.

Naive Set Theory: Any definable collection is a set.
Russell’s Paradox.

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$$\exists y \forall x (x \in y \iff P(x))$$  \hspace{1cm} (1)
Russell’s Paradox.

Naive Set Theory: Any definable collection is a set.

\[ \exists y \forall x (x \in y \iff P(x)) \quad (1) \]

\( y \) is the set of elements that satisfies the proposition \( P(x) \).
Russell’s Paradox.

Naive Set Theory: Any definable collection is a set.

$$\exists y \forall x (x \in y \iff P(x))$$  \hspace{1cm} (1)

$y$ is the set of elements that satisfies the proposition $P(x)$.

$P(x) = x \notin x$.
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\( y \) is the set of elements that satisfies the proposition \( P(x) \).
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There exists a \( y \) that satisfies statement 1 for \( P(\cdot) \).
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Take $x = y$. 

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What type of object is a set that contain sets?
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Axioms changed.
Changing Axioms?

Goedel:
Any set of axioms is either

Changing Axioms?

Goedel:
Any set of axioms is either inconsistent (can prove false statements) or
Changing Axioms?

Goedel:
Any set of axioms is either inconsistent (can prove false statements) or incomplete (true statements cannot be proven.)
Changing Axioms?

Goedel:
Any set of axioms is either inconsistent (can prove false statements) or incomplete (true statements cannot be proven.)

Concrete example:

BTW:
Cantor.. bipolar disorder..
Goedel.. starved himself out of fear of being poisoned..
Russell.. was fine...
..but for two schizophrenic children..

Dangerous work?
See Logicomix by Doxiaidis, Papadimitriou (professor here), Papadatos, Di Donna.
Changing Axioms?

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Concrete example:
Continuum hypothesis: “no cardinality between reals and naturals.”
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See Logicomix by Doxiaidis, Papadimitriou (professor here),
Papadatos, Di Donna.
Is it actually useful?

Write me a program checker!

Check that the compiler works!

Check that the compiler terminates on a certain input.

HALT \((P,I)\)

\(P\) - program
\(I\) - input.

Determines if \(P(I)\) (run on \(I\)) halts or loops forever.

Notice: Need a computer... with the notion of a stored program!!!! (not an adding machine, not a person and an adding machine.)

Program is a text string. Text string can be an input to a program. Program can be an input to a program.
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Is it actually useful?

Write me a program checker!
Check that the compiler works!
How about.. Check that the compiler terminates on a certain input.
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$HALT(P, I)$
Is it actually useful?

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\textit{HALT}(P, I)

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  \item \textit{P} - program
  \item \textit{I} - input.
\end{itemize}

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\[ P \text{ - program} \]
\[ I \text{ - input.} \]

Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

Notice:
Need a computer
...with the notion of a stored program!!!!
Is it actually useful?

Write me a program checker!
Check that the compiler works!
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Program can be an input to a program.
Implementing HALT.

HALT (P, I)
P - program
I - input.
Determines if P(I) (P run on I) halts or loops forever.
Run P on I and check!
How long do you wait?
Something about infinity here, maybe?
Implementing HALT.

$HALT(P, I)$
Implementing HALT.

\[ \text{HALT}(P, I) \]
\[ P - \text{program} \]
Implementing HALT.

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Halt does not exist.

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Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

**Theorem:** There is no program HALT.

**Proof:** Yes! No!
Halt does not exist.

\[ \text{HALT}(P, I) \]
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$\text{HALT}(P, I)$

$P$ - program

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Determines if $P(I)$ ($P$ run on $I$) halts or loops forever.

**Theorem:** There is no program $\text{HALT}$.

**Proof:** Yes! No! Yes! No!

Halt does not exist.

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\[ HALT(P, I) \]

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Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

**Theorem:** There is no program HALT.

**Proof:** Yes! No! Yes! No! No! Yes!
**Halt does not exist.**

\[ HALT(P, I) \]

\begin{itemize}
    \item \textit{P} - program
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\end{itemize}

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HALT does not exist.

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Halt does not exist.

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(A) He is confused.
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Halt does not exist.

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(C) Diagonalization.
Halt does not exist.

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- (A) He is confused.
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(C).
Halt does not exist.

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(C). Maybe (D).
Halt and Turing.

Proof:

Assume there is a program $\text{HALT}(\cdot, \cdot)$. Turing($P$)

1. If $\text{HALT}(P, P) =$ "halts", then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program $\text{HALT}$.

There is text that "is" the program $\text{HALT}$.

There is text that is the program $\text{Turing}$.

Can run Turing on Turing!

Does Turing($\text{Turing}$) halt?

Turing($\text{Turing}$) halts $\Rightarrow$ then $\text{HALTS}(\text{Turing}, \text{Turing}) =$ halts $\Rightarrow$ Turing($\text{Turing}$) loops forever.

Turing($\text{Turing}$) loops forever $\Rightarrow$ then $\text{HALTS}(\text{Turing}, \text{Turing}) \neq$ halts $\Rightarrow$ Turing($\text{Turing}$) halts.

Contradiction.

Program $\text{HALT}$ does not exist!
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$. 

Does $Turing(Turing)$ halt?

- $Turing(Turing)$ halts $\implies$ then $HALT(Turing, Turing) = \text{halts}$ $\implies$ $Turing(Turing)$ loops forever.
- $Turing(Turing)$ loops forever $\implies$ then $HALT(Turing, Turing) \neq \text{halts}$ $\implies$ $Turing(Turing)$ halts.

Contradiction.

Program $HALT$ does not exist!
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot, \cdot)$.

$Turing(P)$
Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Turing($P$)
1. If $HALT(P,P) =$ “halts”, then go into an infinite loop.
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot, \cdot)$.

$Turing(P)$
1. If $HALT(P, P) =$ “halts”, then go into an infinite loop.
2. Otherwise, halt immediately.
Halt and Turing.

**Proof:** Assume there is a program \( HALT(\cdot, \cdot) \).

\( \text{Turing(P)} \)

1. If \( HALT(P,P) = \text{"halts"} \), then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program HALT.
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot, \cdot)$.

$Turing(P)$
1. If $HALT(P,P) =$“halts”, then go into an infinite loop.
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There is text that “is” the program HALT.
Proof: Assume there is a program \( HALT(\cdot,\cdot) \).

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1. If \( HALT(P,P) = \text{"halts"} \), then go into an infinite loop.
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1. If $HALT(P, P) =$ “halts”, then go into an infinite loop.
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Assumption: there is a program $HALT$.
There is text that “is” the program $HALT$.
There is text that is the program Turing.
Can run Turing on Turing!
Halt and Turing.

Proof: Assume there is a program \( \text{HALT}(\cdot, \cdot) \).

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1. If \( \text{HALT}(P, P) = \text{"halts"} \), then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program \( \text{HALT} \).
There is text that “is” the program \( \text{HALT} \).
There is text that is the program \( \text{Turing} \).
Can run \( \text{Turing} \) on \( \text{Turing} \)!

Does \( \text{Turing}(\text{Turing}) \) halt?
Halt and Turing.

Proof: Assume there is a program $HALT(\cdot,\cdot)$.

Turing(P)
1. If $HALT(P,P) = \text{"halts"}$, then go into an infinite loop.
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Assumption: there is a program HALT.
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Can run Turing on Turing!

Does $Turing(Turing)$ halt?

$Turing(Turing)$ halts
Halt and Turing.

**Proof:** Assume there is a program \( HALT(\cdot, \cdot) \).

\[
\text{Turing}(P)
\begin{align*}
1. & \quad \text{If } HALT(P, P) = \text{"halts"}, \text{ then go into an infinite loop.} \\
2. & \quad \text{Otherwise, halt immediately.}
\end{align*}
\]

Assumption: there is a program HALT. There is text that “is” the program HALT. There is text that is the program Turing. Can run Turing on Turing!

Does \( \text{Turing(Turing)} \) halt?

\[
\text{Turing(Turing) halts} \quad \implies \quad \text{then HALTS(Turing, Turing) = halts}
\]
Halt and Turing.

**Proof:** Assume there is a program \( HALT(\cdot, \cdot) \).

\( \text{Turing}(P) \)
1. If \( HALT(P, P) = \text{"halts"} \), then go into an infinite loop.
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There is text that “is” the program \( HALT \).
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Can run \( Turing \) on \( Turing \)!

Does \( Turing(Turing) \) halt?

\( Turing(Turing) \) halts
\[ \Rightarrow \text{then } HALTS(Turing, Turing) = \text{halts} \]
\[ \Rightarrow Turing(Turing) \text{ loops forever.} \]
Halt and Turing.

**Proof:** Assume there is a program \( HALT(\cdot,\cdot) \).

**Turing(P)**
1. If \( HALT(P,P) = \text{"halts"} \), then go into an infinite loop.
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Assumption: there is a program HALT.
There is text that “is” the program HALT.
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Can run Turing on Turing!

Does **Turing(Turing)** halt?

Turing(Turing) halts
\[ \implies \text{then } HALTS(Turing, Turing) = \text{halts} \]
\[ \implies \text{Turing(Turing) loops forever.} \]

Turing(Turing) loops forever
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot, \cdot)$.

$Turing(P)$
1. If $HALT(P, P) \equiv \text{"halts"}$, then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program $HALT$.
There is text that “is” the program $HALT$.
There is text that is the program $Turing$.
Can run $Turing$ on $Turing$!

Does $Turing(Turing)$ halt?

$Turing(Turing)$ halts
\[ \implies \text{then } HALTS(Turing, Turing) = \text{halts} \]
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$Turing(Turing)$ loops forever
\[ \implies \text{then } HALTS(Turing, Turing) \neq \text{halts} \]
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

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1. If $HALT(P,P) = \text{"halts"}$, then go into an infinite loop.
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Turing(Turing) loops forever
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Halt and Turing.

**Proof:** Assume there is a program \(HALT(\cdot, \cdot)\).

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1. If \(HALT(P, P) = \text{"halts"}\), then go into an infinite loop.
2. Otherwise, halt immediately.

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There is text that “is” the program \(HALT\).
There is text that is the program \(Turing\).
Can run Turing on Turing!

Does \(Turing(Turing)\) halt?

\[Turing(Turing) \text{ halts} \implies \text{HALTS(Turing, Turing) }= \text{halts}\]
\[\implies Turing(Turing) \text{ loops forever.}\]

\[Turing(Turing) \text{ loops forever} \implies \text{HALTS(Turing, Turing) }\neq \text{halts}\]
\[\implies Turing(Turing) \text{ halts.}\]

Contradiction.
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot, \cdot)$.

$\text{Turing}(P)$
1. If $HALT(P, P) =$ “halts”, then go into an infinite loop.
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Assumption: there is a program HALT.
There is text that “is” the program HALT.
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Does $\text{Turing}(\text{Turing})$ halt?

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$\implies$ $\text{Turing}(\text{Turing})$ loops forever.

$\text{Turing}(\text{Turing})$ loops forever
$\implies$ then $\text{HALTS}(\text{Turing}, \text{Turing}) \neq \text{halts}$
$\implies$ $\text{Turing}(\text{Turing})$ halts.

Contradiction. Program HALT does not exist!
Halt and Turing.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Turing(P)
1. If $HALT(P, P) =$ “halts”, then go into an infinite loop.
2. Otherwise, halt immediately.

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1. If $HALT(P,P) =$“halts”, then go into an infinite loop.
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Questions?
Another view of proof: diagonalization.

Any program is a fixed length string.
Another view of proof: diagonalization.

Any program is a fixed length string. Fixed length strings are enumerable.
Another view of proof: diagonalization.

Any program is a fixed length string. Fixed length strings are enumerable. Program halts or not any input, which is a string.
Another view of proof: diagonalization.

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Halt - diagonal.
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Halt - diagonal.
Turing - is **not** Halt.
Another view of proof: diagonalization.

Any program is a fixed length string. Fixed length strings are enumerable. Program halts or not any input, which is a string.

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\[
\begin{array}{c|cccc}
 & P_1 & P_2 & P_3 & \ldots \\
\hline 
P_1 & H & H & L & \ldots \\
P_2 & L & L & H & \ldots \\
P_3 & L & H & H & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{array}
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Turing can be constructed from Halt.
Halt does not exist!
Another view of proof: diagonalization.

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Halt - diagonal. Turing - is not Halt. and is different from every $P_i$ on the diagonal. Turing is not on list. Turing is not a program. Turing can be constructed from Halt. Halt does not exist!
Proof play by play.

Assumed $\text{HALT}(P, I)$ existed.
Proof play by play.

Assumed $\text{HALT}(P, I)$ existed.

What is $P$?
Proof play by play.

Assumed \( \text{HALT}(P, I) \) existed.

What is \( P \)? Text.
Proof play by play.

Assumed \( \text{HALT}(P,I) \) existed.

What is \( P \)? Text.
What is \( I \)?
Proof play by play.

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What is \( P \)? Text.
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Assumed $\text{HALT}(P, I)$ existed.

What is $P$? Text.
What is $I$? Text.
Proof play by play.

Assumed $\text{HALT}(P, I)$ existed.

What is $P$? Text.
What is $I$? Text.

What does it mean to have a program $\text{HALT}(P, I)$.
Proof play by play.

Assumed $\text{HALT}(P, I)$ existed.

What is $P$? Text.
What is $I$? Text.

What does it mean to have a program $\text{HALT}(P, I)$.
You have Text that is the program $\text{HALT}(P, I)$. 
Proof play by play.

Assumed \( \text{HALT}(P, I) \) existed.

What is \( P \)? Text.
What is \( I \)? Text.

What does it mean to have a program \( \text{HALT}(P, I) \).
You have \( Text \) that is the program \( \text{HALT}(P, I) \).
Proof play by play.

Assumed $\text{HALT}(P, I)$ existed.

What is $P$? Text.
What is $I$? Text.

What does it mean to have a program $\text{HALT}(P, I)$.
   You have Text that is the program $\text{HALT}(P, I)$.

Have ___ that is the program TURING.
Proof play by play.

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Proof play by play.

Assumed \( \text{HALT}(P, I) \) existed.

What is \( P \)? Text.
What is \( I \)? Text.

What does it mean to have a program \( \text{HALT}(P, I) \).
  You have \text{Text} that is the program \( \text{HALT}(P, I) \).

Have \text{Text} that is the program \text{TURING}.
Here it is!!
Proof play by play.

Assumed $\text{HALT}(P, I)$ existed.

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Here it is!!

$\text{Turing}(P)$
Proof play by play.

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Here it is!!

$\text{Turing}(P)$
1. If $\text{HALT}(P, P) =$“halts”, then go into an infinite loop.
Proof play by play.

Assumed $\text{HALT}(P, I)$ existed.

What is $P$? Text.
What is $I$? Text.

What does it mean to have a program $\text{HALT}(P, I)$.

You have $\text{Text}$ that is the program $\text{HALT}(P, I)$.

Have $\text{Text}$ that is the program $\text{TURING}$.
Here it is!!

$\text{Turing}(P)$
1. If $\text{HALT}(P, P) = \text{"halts"}$, then go into an infinite loop.
2. Otherwise, halt immediately.
Proof play by play.

Assumed HALT\((P,I)\) existed.

What is \(P\)? Text.
What is \(I\)? Text.

What does it mean to have a program HALT\((P,I)\).

You have Text that is the program HALT\((P,I)\).

Have Text that is the program TURING. 
Here it is!!

Turing\((P)\)

1. If HALT\((P,P)\) =“halts”, then go into an infinite loop.
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Turing “diagonalizes” on list of program.
Proof play by play.

Assumed $\text{HALT}(P, I)$ existed.

What is $P$? Text.
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Turing “diagonalizes” on list of program.
It is not a program!!!!
Proof play by play.

Assumed \( \text{HALT}(P, I) \) existed.

What is \( P \)? Text.
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You have Text that is the program \( \text{HALT}(P, I) \).

Have Text that is the program TURING. Here it is!!

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It is not a program!!!!

\( \implies \) HALT is not a program.
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Turing “diagonalizes” on list of program.
It is not a program!!!!
$\implies$ $\text{HALT}$ is not a program.

Questions?
We are so smart!

Wow, that was easy!
We are so smart!

Wow, that was easy!
We should be famous!
No computers for Turing!

In Turing’s time.
No computers for Turing!

In Turing’s time.
No computers.
No computers for Turing!

In Turing’s time.
No computers.
Adding machines.
No computers for Turing!

In Turing’s time.
No computers.
Adding machines.
e.g., Babbage (from table of logarithms) 1812.
No computers for Turing!

In Turing’s time.
No computers.
Adding machines.
e.g., Babbage (from table of logarithms) 1812.
Concept of program as data wasn’t really there.
Turing machine.

Turing machine.

– an (infinite) tape with characters
– be in a state, and read a character
– move left, right, and/or write a character.

Universal Turing machine
– an interpreter program for a Turing machine
– where the tape could be a description of a ...

Turing: AI, self modifying code, learning...
A Turing machine.
– an (infinite) tape with characters
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Now that’s a computer!
A Turing machine.
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Turing: AI, self modifying code, learning...
Turing and computing.

Just a mathematician?
Turing and computing.

Just a mathematician?

“Wrote” a chess program.
Turing and computing.

Just a mathematician?

“Wrote” a chess program.

Simulated the program by hand to play chess.
Just a mathematician?

“Wrote” a chess program.

Simulated the program by hand to play chess.

It won!
Turing and computing.

Just a mathematician?
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It won! Once anyway.
Just a mathematician?

“Wrote” a chess program.
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Involved with computing labs through the 40s.
Church, Gödel and Turing.

Church proved an equivalent theorem. (Previously.)
Church, Gödel and Turing.

Church proved an equivalent theorem. (Previously.)

Used $\lambda$ calculus....
Church, Gödel and Turing.

Church proved an equivalent theorem. (Previously.)
Used $\lambda$ calculus....which is...

Gödel: Incompleteness theorem. Any formal system either is inconsistent or incomplete.
Inconsistent: A false sentence can be proven.
Incomplete: There is no proof for some sentence in the system.
Along the way: "built" computers out of arithmetic.
Showed that every mathematical statement corresponds to.... a natural number!

Today: Programs can be written in ascii.
Church, Gödel and Turing.

Church proved an equivalent theorem. (Previously.)
Used λ calculus....which is... Lisp (Scheme)!!!
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Used $\lambda$ calculus....which is... Lisp (Scheme)!!!

.. functional part.
Church, Gödel and Turing.

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Programming languages!
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Church, Gödel and Turing.

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.... a natural number! ! ! ! Same cardinality as...Text.

Today: Programs can be written in
Church, Gödel and Turing.

Church proved an equivalent theorem. (Previously.)

Used $\lambda$ calculus....which is... Lisp (Scheme)!!!
.. functional part. Scheme’s lambda is calculus’s $\lambda$!

Programming languages! javascript, ruby, python....

Gödel: Incompleteness theorem.

Any formal system either is inconsistent or incomplete.
Inconsistent: A false sentence can be proven.
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Today: Programs can be written in ascii.
Computing on top of computing...

Computer, assembly code, programming language, browser, html, javascript..
Computing on top of computing...

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We can’t get enough of building more Turing machines.
Undecidable problems.

Does a program, $P$, print “Hello World”? 

Find exit points and add statement: Print “Hello World.”

Can a set of notched tiles tile the infinite plane? Proof: simulate a computer. Halts if finite.

Does a set of integer equations have a solution? Example: $x^n + y^n = 1$?

Problem is undecidable. Be careful!

Is there an integer solution to $x^n + y^n = 1$? (Diophantine equation.) The answer is yes or no.

This “problem” is not undecidable. Undecidability for Diophantine set of equations $\Rightarrow$ no program can take any set of integer equations and always correctly output whether it has an integer solution.
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  Fly: blob. Torso becomes striped.
  Developed chemical reaction-diffusion networks that break symmetry.
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- Imitation Game.
Turing: personal.

Tragic ending...
Turing: personal.

Tragic ending...

- Arrested as a homosexual, (not particularly closeted)
Turing: personal.

Tragic ending...

- Arrested as a homosexual, (not particularly closeted)
- given choice of prison or (quackish) injections to eliminate sex drive;

(A bite from the apple....) accident?

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- Arrested as a homosexual, (not particularly closeted)
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- took injections.
Tragic ending...

- Arrested as a homosexual, (not particularly closeted)
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- lost security clearance...

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- Arrested as a homosexual, (not particularly closeted)
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- took injections.
- lost security clearance...
- suffered from depression;
Turing: personal.

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- Given choice of prison or (quackish) injections to eliminate sex drive;
- Took injections.
- Lost security clearance...
- Suffered from depression;
- (Possibly) suicided with cyanide at age 42 in 1954.
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(A British Government apologized (2009) and pardoned (2013).)
Turing: personal.

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*Neither true nor false!*
This statement is a lie. Neither true nor false!

Every person who doesn’t shave themselves is shaved by the barber.
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Program is text, so we can pass it to itself,
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No Turing Program $\implies$ No halt program. ($\neg Q \implies \neg P$)

Program is text, so we can pass it to itself, or refer to self.
Summary: decidability.

Computer Programs are an interesting thing.
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Like Math.
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Like Math. 
Formal Systems.
Computer Programs are an interesting thing. Like Math. Formal Systems.
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  Like Math.
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Computer Programs cannot completely “understand” computer programs.
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Computation is a lens for other action in the world.
What’s to come?
Probability

What’s to come? Probability.
Probability

What’s to come? Probability.

A bag contains:
What’s to come? Probability.

A bag contains:

- Red
- Blue
- Yellow
- Blue
- Red
- Red
- Blue
What’s to come? Probability.

A bag contains:

- 3 red balls
- 2 blue balls
- 1 yellow ball

What is the chance that a ball taken from the bag is blue?
What’s to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?
Count blue.
What’s to come? Probability.

A bag contains:

![Bag with red, blue, and yellow balls](image)

What is the chance that a ball taken from the bag is blue? Count blue. Count total.
What’s to come? Probability.

A bag contains:

- Red
- Blue
- Yellow
- Blue
- Red
- Red
- Red
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What is the chance that a ball taken from the bag is blue?
What’s to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?
For now:
What’s to come? Probability.

A bag contains:

- Red
- Blue
- Yellow
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- Red
- Red
- Red
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For now: Counting!
What’s to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?
For now: Counting!
Later: Probability.