Today.

Farewell to modular arithmetic. Until the midterm. And final.

Coutability and Uncountability.

Undecidability.

Farewell (for now) to modular arithmetic...

Modular arithmetic modulo a prime.

Add, subtract, commutative, associative, inverses!
Allow for solving linear systems, discussing polynomials...

Why not modular arithmetic all the time?

$4 > 3$? Yes!

$4 > 3 \pmod 7$? Yes...maybe?

$-3 > 3 \pmod 7$? Uh oh.. $-3 = 4 \pmod 7$.

Another problem.

4 is close to 3.
But can you get closer? Sure. 3.5. Closer. Sure? 3.25, 3.1, 3.000001. ...

For reals numbers we have the notion of limit, continuity, and derivative.......

....and Calculus.

For modular arithmetic...no Calculus. Sad face!

Next up: how big is infinity.

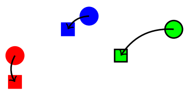
- Countable
- Countably infinite.
- Enumeration

How big are the reals or the integers?

Infinite!

Is one bigger or smaller?

Same size?



Same number?
Make a function f : Circles $\rightarrow$ Squares.
f(red circle) $=$ red square
f(blue circle) $=$ blue square
f(circle with black border) $=$ square with black border
One to one. Each circle mapped to different square.
One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
Onto. Each square mapped to from some circle .
Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

Isomorphism principle: If there is $f : D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.
One to One:
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
or
$\forall x, y \in D$, $f(x) = f(y) \implies x = y$.

Onto: For all $y \in R$, $\exists x \in D, y = f(x)$.

$f(\cdot)$ is a **bijection** if it is one to one and onto.

Isomorphism principle:
If there is a bijection $f : D \rightarrow R$ then $|D| = |R|$.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n, for $z = n+1$, $f(z) = (n+1) - 1 = n$.
Onto for $\mathbb{N}$

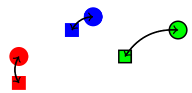
Bijection! $\implies |\mathbb{Z}^+| = |\mathbb{N}|$.

But.. but Where's zero? "Comes from 1."

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.
$f(n) = n+1$. $0 \to 1, 1 \to 2, \ldots$

Bijection from A to B $\implies$ a bijection from B to A.



Inverse function!

Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

More large sets.

E - Even natural numbers?

$f : \mathbb{N} \to E$.

$f(n) \to 2n$.

Onto: $\forall e \in E$, $f(e/2) = e$. $e/2$ is natural since e is even
One-to-one: $\forall x, y \in N, x \neq y \implies 2x \neq 2y$. $\equiv f(x) \neq f(y)$

Evens are countably infinite.
Evens are same size as all natural numbers.

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if x is even and y is even,
then $x/2 \neq y/2 \implies f(x) \neq f(y)$
....

Onto: For any $z \in Z$,
if $z \geq 0$, $f(2z) = z$ and $2z \in N$.
if $z < 0$, $f(2|z| - 1) = z$ and $2|z| + 1 \in N$.

Integers and naturals have same size!

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2
4	2
...	...

Notice that: A listing "is" a bijection with a subset of natural numbers.
Function $\equiv$ "Position in list."
If finite: bijection with $\{0, \ldots, |S| - 1\}$
If infinite: bijection with N.

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",
"Output next element of S"
...
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2, ...\}$
$Z = \{\{0, 1, 2, \ldots, \}$ and then $\{-1, -2, \ldots\}\}$

When do you get to -1? at infinity?

Need to be careful.

61A —- streams!

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,
output only if $x \in T$.

Implications:
Z^+ is countable.
It is infinite since the list goes on.
There is a bijection with the natural numbers.
So it is countably infinite.

All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.
$B = \{0, 1\}^*$.

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$.
ϕ is empty string.

For any string, it appears at some position in the list.
If n bits, it will appear before position 2^{n+1}.

Should be careful here.

$B = \{\phi; , 0, 00, 000, 0000, ...\}$
Never get to 1.

More fractions?

Enumerate the rational numbers in order...

$0, \ldots, 1/2, ..$

Where is $1/2$ in list?

After $1/3$, which is after $1/4$, which is after $1/5$...

A thing about fractions:
any two fractions has another fraction between it.

Can't even get to "next" fraction!

Can't list in "order".

Pairs of natural numbers.

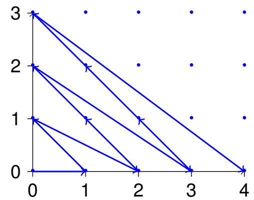
Consider pairs of natural numbers: $N \times N$
E.g.: $(1, 2)$, $(100, 30)$, etc.

For finite sets S_1 and S_2,
then $S_1 \times S_2$
has size $|S_1| \times |S_2|$.

So, $N \times N$ is countably infinite squared ???

Pairs of natural numbers.

Enumerate in list:
$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \ldots...$



The pair (a, b), is in first $\approx (a+b+1)(a+b)/2$ elements of list!
(i.e., "triangle").

Countably infinite.

Same size as the natural numbers!!

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a,b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative ??? No!

Repeatedly and alternatively take one from each list.
  Interleave Streams in 61A

The rationals are countably infinite.

Real numbers..

Real numbers are same size as integers?

The reals.

Are the set of reals countable?

Lets consider the reals $[0,1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$
.632120558... $1 - 1/e$
.345212312... Some real number

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .77677...

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
  and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset $[0,1]$ is not countable!!

All reals?

Subset $[0,1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.

If reals are countable then so is $[0,1]$.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.
4. Show that t is different from all elements in the list
   $\implies$ t is not in the list.
5. Show that t is in S.
6. Contradiction.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}, \{0,\ldots,7\}$, evens, odds, primes,

Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.
otherwise $i \notin D$.

D is different from ith set in L for every i.
$\implies D$ is not in the listing.

D is a subset of N.

L does not contain all subsets of N.

Contradiction.

Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S, is the **powerset** of N.)

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \to [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0,1/2]$ a division $\implies f(x) \neq f(y)$.
If one is in $[0,1/2]$ and one isn't, different ranges $\implies f(x) \neq f(y)$.
Bijection!

$[0,1]$ is same cardinality as nonnegative reals!

Diagonalize Natural Number.

Natural numbers have a listing, L.

Make a diagonal number, D:
differ from ith element of L in ith digit.

Differs from all elements of listing.

D is a natural number... Not.

Any natural number has a finite number of digits.

"Construction" requires an infinite number of digits.

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

First of Hilbert's problems!

Resolution of hypothesis?

Gödel. 1940.
Can't use math!
If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Who shaves the barber?

Self reference.

Can a program refer to a program?

Can a program refer to itself?

Uh oh....

Russell's Paradox.

Naive Set Theory: Any definable collection is a set.

$$\exists y \forall x (x \in y \iff P(x)) \tag{1}$$

y is the set of elements that satifies the proposition $P(x)$.

$P(x) = x \notin x$. Definable set.

There exists a y that satisfies statement 1 for $P(\cdot)$.

Take $x = y$.

$$y \in y \iff y \notin y.$$

Oops! Not Definable.

What type of object is a set that contain sets?

Axioms changed.

Changing Axioms?

Goedel:
Any set of axioms is either
inconsistent (can prove false statements) or
incomplete (true statements cannot be proven.)

Concrete example:
Continuum hypothesis: "no cardinatity between reals and naturals."
Continuum hypothesis not disprovable in ZFC
(Goedel 1940.)

Continuum hypothesis not provable.
(Cohen 1963: only Fields medal in logic)

BTW:
Cantor ..bipolar disorder..
Goedel ..starved himself out of fear of being poisoned..
Russell .. was fine.....but for ...two schizophrenic children..
Dangerous work?

See Logicomix by Doxiaidis, Papadimitriou (professor here), Papadatos, Di Donna.

Is it actually useful?

Write me a program checker!

Check that the compiler works!

How about.. Check that the compiler terminates on a certain input.

$HALT(P, I)$
  P - program
  I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Notice:
Need a computer
...with the notion of a stored program!!!!
(not an adding machine! not a person and an adding machine.)

Program is a text string.
Text string can be an input to a program.
Program can be an input to a program.

Implementing HALT.

$HALT(P, I)$
  P - program
  I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Run P on I and check!

How long do you wait?

Something about infinity here, maybe?

Halt does not exist.

$HALT(P, I)$
  P - program
  I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Theorem: There is no program HALT.

Proof: Yes! No! Yes! No! No! Yes! No! Yes! .. $\square$

What is he talking about?
(A) He is confused.
(B) Fermat's Theorem.
(C) Diagonalization.
(D) Professor is just strange.

(C). Maybe (D).

Halt and Turing.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Turing(P)
1. If HALT(P,P) ="halts", then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program HALT.
There is text that "is" the program HALT.
There is text that is the program Turing.
Can run Turing on Turing!

Does Turing(Turing) halt?

Turing(Turing) halts
$\Longrightarrow$ then HALTS(Turing, Turing) $=$ halts
$\Longrightarrow$ Turing(Turing) loops forever.

Turing(Turing) loops forever
$\Longrightarrow$ then HALTS(Turing, Turing) $\neq$ halts
$\Longrightarrow$ Turing(Turing) halts.

Contradiction. Program HALT does not exist! $\square$
Questions?

Another view of proof: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.
Program halts or not any input, which is a string.

	P_1	P_2	P_3	$\cdots$
P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Halt - diagonal.
Turing - is not Halt.
and is different from every P_i on the diagonal.
Turing is not on list. Turing is not a program.
Turing can be constructed from Halt.
Halt does not exist! $\square$

Proof play by play.

Assumed HALT(P, I) existed.

What is P? Text.
What is I? Text.

What does it mean to have a program HALT(P, I).
You have *Text* that is the program HALT(P, I).

Have Text that is the program TURING.
Here it is!!
Turing(P)

1. If HALT(P,P) =“halts”, then go into an infinite loop.
2. Otherwise, halt immediately.

Turing “diagonalizes” on list of program.
It is not a program!!!!
$\implies$ HALT is not a program.

Questions?

We are so smart!

Wow, that was easy!

We should be famous!

No computers for Turing!

In Turing’s time.

No computers.

Adding machines.
e.g., Babbage (from table of logarithms) 1812.

Concept of program as data wasn’t really there.

Turing machine.

A Turing machine.
– an (infinite) tape with characters
– be in a state, and read a character
– move left, right, and/or write a character.

Universal Turing machine
– an interpreter program for a Turing machine
– where the tape could be a description of a ... Turing machine!

Now that’s a computer!

Turing: AI, self modifying code, learning...

Turing and computing.

Just a mathematician?

“Wrote” a chess program.

Simulated the program by hand to play chess.

It won! Once anyway.

Involved with computing labs through the 40s.

Church, Gödel and Turing.

Church proved an equivalent theorem. (Previously.)

Used λ calculus....which is... Lisp (Scheme)!!!
.. functional part. Scheme's lambda is calculus's λ!

Programming languages! javascript, ruby, python....

Gödel: Incompleteness theorem.

Any formal system either is inconsistent or incomplete.
Inconsistent: A false sentence can be proven.
Incomplete: There is no proof for some sentence in the system.

Along the way: "built" computers out of arithmetic.
Showed that every mathematical statement corresponds to
.... a natural number! ! ! ! Same cardinality as...Text.
Today:Programs can be written in ascii.

Computing on top of computing...

Computer, assembly code, programming language, browser, html, javascript..

We can't get enough of building more Turing machines.

Undecidable problems.

Does a program, P, print "Hello World"?
How? What is P? Text!!!!!!

Find exit points and add statement: **Print** "Hello World."

Can a set of notched tiles tile the infinite plane?
Proof: simulate a computer. Halts if finite.

Does a set of integer equations have a solution?
Example: " $x^n + y^n = 1$?"
Problem is undecidable.

Be careful!

Is there an integer solution to $x^n + y^n = 1$?
(Diophantine equation.)

The answer is yes or no. This "problem" is not undecidable.

Undecidability for Diophantine set of equations
$\implies$ no program can take any set of integer equations and
always corectly output whether it has an integer solution.

More about Alan Turing.

- Brilliant codebreaker during WWII, helped break German Enigma Code (which probably shortened war by 1 year).
- Seminal paper in numerical analysis: Condition number.
  Math 54 doesn't really work.
  Almost dependent matrices.
- Seminal paper in mathematical biology.
  Person: embryo is blob. Legs, arms, head.... How?
  Fly: blob. Torso becomes striped.
  Developed chemical reaction-diffusion networks that break symmetry.
- Imitation Game.

Turing: personal.

Tragic ending...

- Arrested as a homosexual, (not particularly closeted)
- given choice of prison or (quackish) injections to eliminate sex drive;
- took injections.
- lost security clearance...
- suffered from depression;
- (possibly) suicided with cyanide at age 42 in 1954.
  (A bite from the apple....) accident?
- British Government apologized (2009) and pardoned (2013).

Back to technical..

This statement is a lie. Neither true nor false!

Every person who doesn't shave themselves is shaved by the barber.

Who shaves the barber?

```
def Turing(P):
 if Halts(P,P): while(true): pass
 else:
  return
```

...Text of Halt...

Halt Progam $\implies$ Turing Program. ($P \implies Q$)

Turing("Turing")? Neither halts nor loops! $\implies$ No Turing program.

No Turing Program $\implies$ No halt program. ($\neg Q \implies \neg P$)

Program is text, so we can pass it to itself,
or refer to self.

Summary: decidability.

Computer Programs are an interesting thing.
  Like Math.
  Formal Systems.

Computer Programs cannot completely "understand" computer programs.

Computation is a lens for other action in the world.

Probability

What's to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total. Divide.

For now: Counting!

Later: Probability.