CS 70 Spring 2018 — Discrete Mathematics and Probability Theory — Babak Ayazifar and Satish Rao — HW 13

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

1 LLSE and Graphs

Consider a graph with n vertices numbered 1 through n, where n is a positive integer ≥ 2. For each pair of distinct vertices, we add an undirected edge between them independently with probability p. Let D_1 be the random variable representing the degree of vertex 1, and let D_2 be the random variable representing the degree of vertex 2.

(a) Compute $\mathbb{E}[D_1]$ and $\mathbb{E}[D_2]$.

(b) Compute $\text{var}(D_1)$.

(c) Compute $\text{cov}(D_1, D_2)$.

(d) Using the information from the first three parts, what is $L(D_2 \mid D_1)$?

2 Swimsuit Season

In the swimsuit industry, it is well-known that there is a “swimsuit season”. During this time, swimsuit sales skyrocket!

We will model this with a random variable X which is either λ_L or λ_H with equal probability; λ_L represents the mean number of customers in a day when swimsuits are not in season, and λ_H represents the mean number of customers during swimsuit season. So, λ_L is the "low rate" and λ_H is the "high rate". The number of customer arrivals Y on a particular day is modeled as a Poisson random variable with mean X.

You observe Y customers on a certain day, and the task is to estimate X.

(a) What is $L[X \mid Y]$?

(b) What is $\mathbb{E}[X \mid Y]$?

3 Quadratic Regression

In this question, we will find the best quadratic estimator of Y given X. First, some notation: let μ_i be the ith moment of X, i.e. $\mu_i = \mathbb{E}[X^i]$. Also, define $\beta_1 = \mathbb{E}[XY]$ and $\beta_2 = \mathbb{E}[X^2Y]$. For simplicity, we will assume that $\mathbb{E}[X] = \mathbb{E}[Y] = 0$ and $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1$. (Note that this poses no loss of generality, because we can always transform the random variables by subtracting their means and dividing by their standard deviations.) We claim that the best quadratic estimator of Y given X is

$$\hat{Y} = \frac{1}{\mu_3^2 - \mu_4 + 1}(aX^2 + bX + c)$$

where

$$\begin{aligned}
a &= \mu_3\beta_1 - \beta_2, \\
b &= (1-\mu_4)\beta_1 + \mu_3\beta_2, \\
c &= -\mu_3\beta_1 + \beta_2.
\end{aligned}$$

Your task is to prove the Projection Property for $\hat{Y}$.

(a) Prove that $\mathbb{E}[Y - \hat{Y}] = 0$.

(b) Prove that $\mathbb{E}[(Y - \hat{Y})X] = 0$.

(c) Prove that $\mathbb{E}[(Y - \hat{Y})X^2] = 0$.

Any quadratic function of X is a linear combination of 1, X, and X^2. Hence, these equations together imply that $Y - \hat{Y}$ is orthogonal to any quadratic function of X, and so $\hat{Y}$ is the best quadratic estimator of Y.

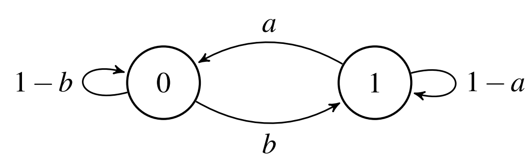
4 Marbles in a Bag

We have r red marbles, b blue marbles, and g green marbles in the same bag. If we sample marbles with replacement until we get 3 red marbles (not necessarily consecutively), how many blue marbles should we expect to see? (*Hint*: It might be useful to use Law of Total Expectation, $E(Y) = E(E(Y|X))$)

5 Markov Chain Terminology

In this question, we will walk you through terms related to Markov chains.

1. (Irreducibility) A Markov chain is irreducible if, starting from any state i, the chain can transition to any other state j, possibly in multiple steps.
2. (Periodicity) $d(i) := \gcd\{n > 0 \mid P^n(i,i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}$, $i \in \mathscr{X}$. If $d(i) = 1\ \forall i \in \mathscr{X}$, then the Markov chain is aperiodic; otherwise it is periodic.
3. (Matrix Representation) Define the transition probability matrix P by filling entry (i,j) with probability $P(i,j)$.
4. (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equations: $\pi = \pi P$.



(a) For what values of a and b is the above Markov chain irreducible? Reducible?

(b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.

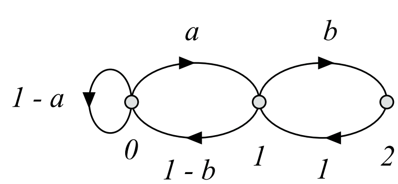
(c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.

(d) Construct a transition probability matrix using the above Markov chain.

(e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

6 Analyze a Markov Chain

Consider the Markov chain $X(n)$ with the state diagram shown below where $a, b \in (0,1)$.



(a) Show that this Markov chain is aperiodic;

(b) Calculate $\mathbb{P}[X(1) = 1, X(2) = 0, X(3) = 0, X(4) = 1 \mid X(0) = 0]$;

(c) Calculate the invariant distribution;

(d) Let $T_i = \min\{n \geq 0 \mid X(n) = i\}$, T_i is the number of steps until we transit to state i for the first time. Calculate $\mathbb{E}[T_2 \mid X(0) = 1]$.