CS 70 Spring 2018 — Discrete Mathematics and Probability Theory — Babak Ayazifar and Satish Rao — HW 10

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

1 Fundamentals

True or false? For the following statements, provide either a proof or a simple counterexample. Let X,Y,Z be arbitrary random variables.

(a) If (X,Y) are independent and (Y,Z) are independent, then (X,Z) are independent.

(b) If (X,Y) are dependent and (Y,Z) are dependent, then (X,Z) are dependent.

(c) Assume X is discrete. If $\text{var}(X) = 0$, then X is a constant.

(d) $\mathbb{E}[X]^4 \leq \mathbb{E}[X^4]$

2 Balls and Bins

Throw n balls into m bins, where m and n are positive integers. Let X be the number of bins with exactly one ball. Compute $\text{var}X$.

3 Proof with Indicators

Let $n \in \mathbb{Z}_+$. Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and let $A_1, \dots, A_n$ be events. Prove that $\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \mathbb{P}(A_i \cap A_j) \geq 0$. Note that α_i can be less than 0.

4 Boutique Store

(a) Consider a boutique store in a busy shopping mall. Every hour, a large number of people visit the mall, and each independently enters the boutique store with some small probability. The store owner decides to model X, the number of customers that enter her store during a particular hour, as a Poisson random variable with mean λ. Suppose that whenever a customer enters the boutique store, they leave the shop without buying anything with probability p. Assume that customers act independently, i.e. you can assume that they each simply flip a biased coin to decide whether to buy anything at all. Let us denote the number of customers that buy something as Y and the number of them that do not buy anything as Z (so $X = Y + Z$). What is the probability that $Y = k$ for a given k? How about $\mathbb{P}[Z = k]$? Prove that Y and Z are Poisson random variables themselves.

Hint: You can use the identity

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

(b) Prove that Y and Z are independent.

5 Ordering Random Variables

Here we will investigate the properties of *ordered collections* of identically distributed (not identical!) random variables. The techniques in this problem can be applied to any kind of distribution, but here we will consider a specific case. Let $X_1, X_2, \dots, X_n$ be iid geometric with parameter p.

(a) Find $\mathbb{P}[X_i > k]$ and $\mathbb{P}[X_i < k]$ for all i and k

(b) Let $X_{(1)}, X_{(2)}, \dots X_{(n)}$ be the random variables corresponding to the *ordered* collection from above. In other words, consider the set of random variables $S = \{X_i \mid 1 \leq i \leq n\}$, and let $X_{(j)}$ be the jth smallest element in that set.

What is $X_{(1)}$, as a function of $X_1 \dots X_n$? (We're just looking to see that you understand the definition. No work necessary)

(c) Find $\mathbb{P}[X_{(1)} \leq k]$.

(d) Find $\mathbb{P}[X_{(i)} \leq k]$.

Hint: There are no tricks to simplify it, unlike the last case. You will end up with a sum. First try to translate "$X_{(i)} \leq k$" into a statement about all of $X_{(1)}$ through $X_{(n)}$ (what do we know about each if $X_{(i)} \leq k$?). Then relate this statement to events involving $X_1, \dots X_n$.

(e) Calculate $\mathbb{P}[X_{(1)} = k_1, X_{(2)} = k_2, \dots, X_{(n)} = k_n]$ using symmetry arguments, assuming all the k_i are distinct. That is, assume $k_1 \neq k_2 \cdots \neq k_n$.

(f) The probabilities in the previous part are associated with the *joint distribution* of $X_{(1)}, X_{(2)}, \dots, X_{(n)}$. If we want to completely specify the joint distribution, we cannot limit ourselves to only cases where the k_i are distinct.

How would you modify your calculation to account for the possibility that not all the k_i are distinct? Either an explanation in words or an explicit calculation is fine.

6 Geometric and Poisson

Let X be geometric with parameter p, Y be Poisson with parameter λ, and $Z = \max(X, Y)$. For each of the following parts, your final answers should not have summations.

(a) Compute $P(X > Y)$.

(b) Compute $P(Z \geq X)$.

(c) Compute $P(Z \leq Y)$.