

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

1 Indicator Variables

- (a) After throwing n balls into m bins at random, what is the expected number of bins that contains exactly k balls?
- (b) Alice and Bob each draw k cards out of a deck of 52 distinct cards with replacement. Find k such that the expected number of common cards that both Alice and Bob draw is at least 1.
- (c) How many people do you need in a room so that you expect that there is going to be a shared birthday on a Monday of the year (assume 52 Mondays in a year and 365 days in a year)?

2 Sinho's Dice

Sinho rolls three fair-sided dice.

- (a) Let X denote the maximum of the three values rolled. What is the distribution of X (that is, $\mathbb{P}[X = x]$ for $x = 1, 2, 3, 4, 6$)? You can leave your final answer in terms of "x". [Hint: Let X denote the maximum of the three values roled. First compute $\mathbb{P}[X \leq x]$ for $x = 1, 2, 3, 4, 5, 6$].
- (b) Let Y denote the minimum of the three values rolled. What is the distribution of Y ?

3 Linearity

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

- (a) In an arcade, you play game A 10 times and game B 20 times. Each time you play game A , you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?
- (b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence “book” appears?
- (c) A building has n floors numbered $1, 2, \dots, n$, plus a ground floor G . At the ground floor, m people get on the elevator together, and each gets off at a uniformly random one of the n floors (independently of everybody else). What is the expected number of floors the elevator stops at (not counting the ground floor)?
- (d) A coin with heads probability p is flipped n times. A “run” is a maximal sequence of consecutive flips that are all the same. (Thus, for example, the sequence $HTHHHTTH$ with $n = 8$ has five runs.) Show that the expected number of runs is $1 + 2(n - 1)p(1 - p)$. Justify your calculation carefully.

4 Random Coloring

Consider a graph $G(V, E)$ with m edges and a set of q colors. Our goal is to prove that there exists a vertex-coloring of the graph using these q colors so that at most $\frac{m}{q}$ edges are monochromatic. An edge is monochromatic if both its vertices are assigned the same color.

- (a) Suppose we color the graph randomly. That is, for each vertex $v \in V$ we choose a color uniformly at random and assign it to v . Prove that the expected number of monochromatic edges is $\frac{m}{q}$.
- (b) Conclude that there exists a vertex-coloring of G so that at most $\frac{m}{q}$ edges are monochromatic. (Hint: Suppose that every coloring of G is such so that at least $\frac{m}{q} + 1$ edges are monochromatic and reach a contradiction.)

5 Long Streaks

Suppose we flip a coin n times to obtain a sequence of flips X_1, X_2, \dots, X_n . A *streak* of flips is a consecutive subsequence of flips that are all the same. For example, if X_3, X_4 , and X_5 are all heads, there is a streak of length 3 starting at the third flip.

- (a) Let n be a power of 2. Show that the expected number of streaks of length $\log_2 n + 1$ is $1 - \frac{\log_2 n}{n}$ and notice that for large n it tends to 1.
- (b) Given a sequence of flips of length n , suppose we break it up into disjoint blocks of $\lfloor \log_2 n - 2 \log_2 \log_2 n \rfloor$ consecutive flips. Show that for every block the probability that is a streak is at least $2 \frac{(\log_2 n)^2}{n}$.
- (c) Show that, for sufficiently large n , the probability that there is no streak of length at least $\lfloor \log_2 n - 2 \log_2 \log_2 n \rfloor$ is less than $1/n$. (Hint: Use the previous part. You might find useful the fact that for every $x > 0$ we have $(1 - \frac{1}{x}) < e^{-x}$).