

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

1 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. For this problem, you do not need to show work that justifies your answers. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

- (a) How many ways are there to arrange n 1s and k 0s into a sequence?
- (b) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.
How many different 13-card bridge hands are there? How many different 13-card bridge hands are there that contain no aces? How many different 13-card bridge hands are there that contain all four aces? How many different 13-card bridge hands are there that contain exactly 6 spades?
- (c) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?
- (d) How many 99-bit strings are there that contain more ones than zeros?
- (e) An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an

English word.

How many different anagrams of FLORIDA are there? How many different anagrams of ALASKA are there? How many different anagrams of ALABAMA are there? How many different anagrams of MONTANA are there?

- (f) How many different anagrams of ABCDEF are there if: (1) C is the left neighbor of E; (2) C is on the left of E.
- (g) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).
- (h) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).
- (i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).
- (j) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?
- (k) How many solutions does $x_0 + x_1 + \dots + x_k = n$ have, if each x must be a non-negative integer?
- (l) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?
- (m) How many solutions does $x_0 + x_1 + \dots + x_k = n$ have, if each x must be a *strictly positive* integer?

2 N-Bit

Let *NBit* be a program that takes in a program P and an input x and outputs “true” if the program ever directly assigns a variable to an n -bit number and “false” otherwise. Can *NBit* exist? If so, describe the procedure, and if not, prove that it cannot exist. (*Hint: model off of the proof that TestHalt cannot exist*)

3 Counting Cartesian Products

For two sets A and B , define the Cartesian Product as $A \times B = \{(a, b) : a \in A, b \in B\}$.

- (a) Given two countable sets A and B , prove that $A \times B$ is countable.
- (b) Given a finite number of countable sets A_1, A_2, \dots, A_n , prove that $A_1 \times A_2 \times \dots \times A_n$ is countable.
- (c) Consider an infinite number of countable sets: B_1, B_2, \dots . Under what condition(s) is $B_1 \times B_2 \times \dots$ countable? Prove that if this condition is violated, $B_1 \times B_2 \times \dots$ is uncountable.

4 Vacation Time

After a number of complaints, the Dunder Mifflin Paper Company has decided on the following rule for vacation leave for the next year (365 days): Every employee must take exactly one vacation leave of 4 consecutive days, one vacation leave of 5 consecutive days and one vacation leave of 6 consecutive days within the year, with the property that any two of the vacation leaves have a gap of at least 7 days between them. In how many ways can an employee arrange their vacation time? (The vacation policy resets every year, so there is no need to worry about leaving a gap between this year and next year's vacations).

5 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

- (a) We place p beads onto a string. How many different ways are there to construct such a sequence of p beads with up to k different colors?
- (b) How many sequences of p beads on the string are there that use at least two colors?
- (c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGGB, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p .)

[*Hint:* Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

- (d) Use your answer to part (c) to prove Fermat's little theorem.

6 Story Problems

Prove the following identities by combinatorial argument:

(a) $\binom{2n}{2} = 2\binom{n}{2} + n^2$

(b) $n^2 = 2\binom{n}{2} + n$

(c) $\sum_{k=0}^n k\binom{n}{k} = n2^{n-1}$

Hint: Consider how many ways there are to pick groups of people ("teams") and then a representative ("team leaders").

(d) $\sum_{k=j}^n \binom{n}{k} \binom{k}{j} = 2^{n-j} \binom{n}{j}$

Hint: Consider a generalization of the previous part.