CS 70 Discrete Mathematics and Probability Theory
Spring 2018 Satish Rao and Babak Ayazifar

HW 1

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

1 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $(\forall x \in \mathbb{R})\ (x \in \mathbb{C})$

(e) $(\forall x \in \mathbb{Z})\ ((2 \mid x \vee 3 \mid x) \implies 6 \mid x)$

(f) $(\forall x \in \mathbb{N})\ ((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

2 Miscellaneous Logic

(a) Let the statement, $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}\ G(x,y)$, be true for predicate $G(x,y)$.

For each of the following statements, decide if the statement is certainly true, certainly false, or possibly true, and justify your solution.

(i) $G(3,4)$

(ii) $\forall x \in \mathbb{R}\ G(x,3)$

(iii) $\exists y\ G(3,y)$

(iv) $\forall y\ \neg G(3,y)$

(v) $\exists x\ G(x,4)$

(b) Give an expression using terms involving $\vee, \wedge$ and $\neg$ which is true if and only if exactly one of X, Y, and Z are true. (Just to remind you: $(X \wedge Y \wedge Z)$ means all three of X,Y,Z are true, $(X \vee Y \vee Z)$ means at least one of X,Y and Z is true.)

3 Prove or Disprove

(a) $\forall n \in \mathbb{N}$, if n is odd then $n^2 + 2n$ is odd.

(b) $\forall x, y \in \mathbb{R}$, $\min(x,y) = (x + y - |x - y|)/2$.

(c) $\forall a, b \in \mathbb{R}$ if $a + b \leq 10$ then $a \leq 7$ or $b \leq 3$.

(d) $\forall r \in \mathbb{R}$, if r is irrational then $r + 1$ is irrational.

(e) $\forall n \in \mathbb{N}$, $10n^2 > n!$.

4 Preserving Set Operations

Define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Hint: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $\forall x,\ x \in X \implies x \in Y$.

1. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
2. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.
3. $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.
4. $f(A \cup B) = f(A) \cup f(B)$.
5. $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.
6. $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

5 Hit or Miss?

State which of the proofs below is correct or incorrect. For the incorrect ones, please explain clearly where the logical error in the proof lies. Simply saying that the claim or the induction hypothesis is false is *not* a valid explanation of what is wrong with the proof. You do not need to elaborate if you think the proof is correct.

(a) **Claim:** For all positive numbers $n \in \mathbb{R}$, $n^2 \geq n$.

Proof. The proof will be by induction on n.
Base Case: $1^2 \geq 1$. It is true for $n = 1$.
Inductive Hypothesis: Assume that $n^2 \geq n$.
Inductive Step: We must prove that $(n+1)^2 \geq n+1$. Starting from the left hand side,

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &\geq n + 1.\end{aligned}$$

Therefore, the statement is true. □

(b) **Claim:** For all negative integers n, $-1 - 3 - \cdots + (2n+1) = -n^2$.

Proof. The proof will be by induction on n.
Base Case: $-1 = -(-1)^2$. It is true for $n = -1$.
Inductive Hypothesis: Assume that $-1 - 3 - \cdots + (2n+1) = -n^2$.
Inductive Step: We need to prove that the statement is also true for $n-1$ if it is true for n, that is, $-1 - 3 - \cdots + (2(n-1)+1) = -(n-1)^2$. Starting from the left hand side,

$$\begin{aligned}-1 - 3 - \cdots + (2(n-1)+1) &= (-1 - 3 - \cdots + (2n+1)) + (2(n-1)+1) \\ &= -n^2 + (2(n-1)+1) \quad \text{(Inductive Hypothesis)} \\ &= -n^2 + 2n - 1 \\ &= -(n-1)^2.\end{aligned}$$

Therefore, the statement is true. □

(c) **Claim:** For all nonnegative integers n, $2n = 0$.

Proof. We will prove by strong induction on n.
Base Case: $2 \times 0 = 0$. It is true for $n = 0$.
Inductive Hypothesis: Assume that $2k = 0$ for all $0 \leq k \leq n$.
Inductive Step: We must show that $2(n+1) = 0$. Write $n+1 = a+b$ where $0 < a, b \leq n$. From the inductive hypothesis, we know $2a = 0$ and $2b = 0$, therefore,

$$2(n+1) = 2(a+b) = 2a + 2b = 0 + 0 = 0.$$

The statement is true. □

6 Badminton Ranking

A team of n ($n \geq 2$) badminton players held a tournament, where every person plays with every other person exactly once, and there are no ties. Prove by induction that after the tournament, we can arrange the n players in a sequence, so that every player in the sequence has won against the person immediately to the right of him.